Appendix B
Evaluating the Reliability of Existing Levees

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Preface

This report is a product of the U. S. Army Corps of Engineers' **Risk Analysis for Water Resources Investments Research Program** managed by the Institute for Water Resources (IWR), Water Resources Support Center (WRC). The work was performed under Work Unit 32835, "Risk Analysis for Stability Evaluation of Levees." Dr. Edward B. Perry of the U.S. Army Engineer Waterways Experiment Station (WES) managed the work unit and Dr. David A. Moser of IWR manages the Risk Analysis Program. Dr. Perry works under the direct supervision of Mr. W. Milton Myers, Chief, Soil Mechanics Branch, Soil and Rock Mechanics Division (S&RMD), Geotechnical Laboratory (GL), and the general supervision of Dr. Don C. Banks, Chief, S&RMD, and Dr. William F. Marcuson III, Director, GL, WES. Dr. Moser works under the direct supervision of Mr. Michael R. Krouse, Chief of the Technical Analysis and Research Division and the general supervision of Mr. Kyle E. Shilling, Director of the IWR.

Mr. Robert Daniel, Chief, Plan Formulation and Evaluation Branch, Policy and Planning Division; Mr. Earl Eiker, Chief, Hydrology and Hydraulics Branch, Engineering Division; and Mr. James E. Crews, Deputy Chief, Operations, Construction and Readiness Division; all within the Civil Works Directorate, Headquarters, U.S. Army Corps of Engineers, serve as Technical Monitors for the Risk Analysis Program.

This report was written by Dr. Thomas F. Wolff, Associate Professor, Department of Civil and Environmental Engineering, Michigan State University, under Contract No. DACW39-94-M-4226 to WES. Dr. Wolff was assisted in the work by Dr. Mostafa Ashoor, Ms. Cynthia Ramon, and Mr. Todd Richter.

At the time of publication of this report, Director of WES was Dr. Robert W. Whalin. Commander was COL Bruce K. Howard, EN. Mr. Kyle E. Schilling was Acting Director of the WRC.

Note that, in Chapter 2, the comments on the rescinded Corps document that appeared in the original report have been edited and deleted during preparation of this ETL.

Conversion Factors, Non-SI to SI Units of Measurement

Non-SI units of measurement used in this report can be converted to SI units as follows:

Multiply	By	To Obtain
degrees	0.0174533	radians
feet	0.3048	meters
inches	2.54	centimeters
feet per second	30.48	centimeters per second
pounds (force) per square foot	0.04788	kilopascals
pounds (force) per square foot	478.802631	dynes per square centimenter
pounds (mass) per cubic foot	0.1570873	kilonewtons per cubic meter

1 Introduction

Purpose

The purpose of the research effort leading to this report was to develop, test, and illustrate procedures that can be used by geotechnical engineers to assign conditional probabilities of failure for existing levees as functions of floodwater elevation. Such functions are in turn to be used by economists when estimating benefits to be derived from proposed levee improvements.

Limitations of Engineering Reliability Analysis

Accuracy of probabilistic measures

Before proceeding, it is important to define a context in which to place engineering reliability analysis and its relationship to flood control levees. The application of probabilistic analysis in geotechnical engineering and other areas of civil engineering is still an emerging technology. Much experience with such procedures remains to be gained, and the appropriate form and shape of probability distributions for the relevant parameters are not known with certainty. The methods described herein should not be expected to provide “true,” or "absolute" probability-of-failure values but can provide consistent measures of *relative reliability* when reasonable assumptions are employed. Such comparative measures can be used to indicate, for example, which reach (or length) of levee, which typical section, or which alternative design may be more reliable than another. They also can be used to determine which of several performance modes (seepage, slope stability, etc.) governs the reliability of a particular levee. All of the levee reaches analyzed are considered independent and unrelated.

Calibration of procedures

Any reliability-based evaluation must be *calibrated*; i.e., tested against a sufficient number of well-understood engineering problems to ensure that it provides reasonable results. Performance modes known to be problematical (such as seepage) should be found to have a lower reliability than those for which problems are seldom observed; larger and more stable sections should be found to be more reliable than smaller, less stable sections, etc. This study provides a beginning point on such calibration studies by performing example analyses on two hypothetical levee sections. As additional analyses are performed, by both researchers and practitioners, on a wide range of real levee cross sections using real data, it is inevitable that adjustments and refinements in the procedures will be required.

Application to economic analysis

When the developed functions are used in an economic analysis, one may perceive a greater degree of precision than really exists, not unlike long-term projections of uncertain costs and benefits. Users are cautioned that functions developed using the presented methods still retain some inherent uncertainty in the absolute sense. Nevertheless, they also contain more information than deterministic approaches to the same problem. The use of a consistent probabilistic framework, with personal judgment checks for reasonableness, should have the advantage and appeal of consistency when compared to the alternative method of trying to identify a single flood elevation at which a levee changes from being reliable to unreliable.

Background

When the Corps of Engineers proposes construction of new flood control levees or improvement of existing levees (typically by raising the height), economic studies are required to assess the relative benefits and costs of the work. Where an existing levee is already present, the project benefits accrue from a difference in the degree of protection. Economic assessment of the levee improvement in turn requires an engineering determination of the probable level of protection afforded by the existing levee.

Past practice

In the past, existing levees that had not been designed or constructed to Corps of Engineers' standards were sometimes, if not often, taken to be nonexistent in economic analysis or taken to afford protection to some low and rather arbitrary elevation. This is no longer permitted; cost-benefit studies for water resource projects are increasingly being cast in a probabilistic framework wherein it is recognized that neither costs nor benefits have precise, predictable values, but rather can assume a range of values associated with a range of likelihoods. Hence, an existing levee is considered to afford protection with some associated probability.

Current practice for navigation rehabilitation studies

For similar economic studies involving the rehabilitation of Corps' navigation locks and dams, possible adverse events that would demand expenditures (e.g. sliding of a lock monolith that would impede navigation) are now analyzed in a probabilistic framework. Investments in rehabilitation work to forestall adverse structural performance are evaluated based on the reliability of components, the probability of adverse performance, and the probable cost of the consequences. Several studies have been conducted to develop procedures (Wolff and Wang 1992a, 1992b; Shannon and Wilson, Inc., and Wolff 1994) and to promulgate guidance (ETL 1110-2-532, U.S. Army Corps of Engineers 1992) for probabilistic analysis of hydraulic structures.

The Conditional Probability of Failure Function

For an existing levee subjected to a flood, the probability of failure P_f can be expressed as a function of the floodwater elevation and other factors including flood duration, soil strength, permeability, embankment geometry, foundation stratigraphy, etc. This study will focus on developing the *conditional* probability of failure function for the floodwater elevation, which will be constructed using engineering estimates of the probability functions or moments of the other relevant variables.

The conditional probability of failure can be written as:

$$Pr_f = Pr(failure|FWE) = f(FWE, X_1, X_2, ..X_n) \qquad (1)$$

In the above expression, the first term (denoting probability of failure) will be used as a shorthand version of the second term. In the second term, the symbol "|" is read *given* and the variable *FWE* is the floodwater elevation. In the third term, the random variables X_1 through X_n denote relevant parameters such as soil strength, permeability, top stratum thickness, etc. Equation 1 can be restated as follows: "The probability of failure, given the floodwater elevation, is a function of the floodwater elevation and other random variables."

Two extreme values of the function can be readily estimated by engineering judgment:

a. For floodwater at the same level as the landside toe (base elevation) of the levee, $P_f = 0$.

b. For floodwater at or near the levee crown (top elevation), $P_f \rightarrow 1.00$.

It may be argued that the probability of failure value may be something less than 1.0 with water at the crown, as additional protection can be provided by emergency measures. The question of primary economic interest, however, is the shape of the function between these extremes. Quantifying this shape is the focus of this study; how reliable might the levee be for, say, a 10- or 20-year flood event that reaches half or three-quarters the height of the levee?

Reliability (R) is defined as:

$$R = 1 - P_f \tag{2}$$

hence, for any floodwater elevation, the probability of failure and reliability must sum to unity.

For the case of floodwater partway up a levee, R could be very near zero or very near unity, depending on engineering factors such as levee geometry, soil strength and permeability, foundation stratigraphy, etc. In turn, these differences in the conditional reliability function could result in very different economic scenarios. Four possible shapes of the reliability versus floodwater elevation are illustrated in Figure 1.

As illustrated by these example curves, the conditional probability of failure function could have a wide range of shapes. For a "good" levee, the probability of failure may remain low and the reliability remain high until the floodwater elevation is rather high. In contrast, a "poor" levee may experience greatly reduced reliability when subjected to even a small flood head. It is hypothesized that some real levees may follow the intermediate curve, which is similar in shape to the "good" case for small floods, but reverses to approach the "poor" case for floods of significant height. Finally, a straight line function is shown in Figure 1, representing a linear relation between reliability and flood height. Although such a linear approximation is shown in current Corps guidance (Policy Guidance Letter No. 26, U.S. Army Corps of Engineers 1991), linearity would not be expected to be the general case.

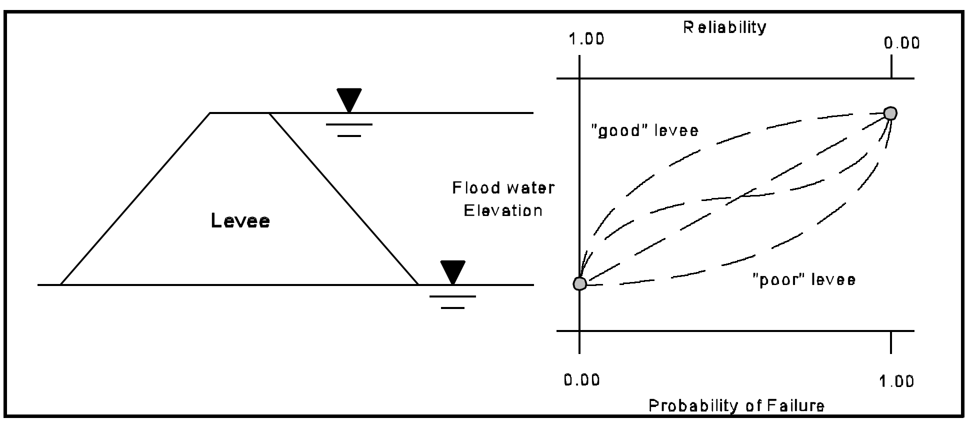


Figure 1. Possible reliability versus floodwater elevation functions for existing levees

Study Approach

To assess the differences in benefits between an existing levee and a proposed improved levee, an economist desires the engineering assessment of the levee reliability quantified in a probabilistic form such as Figure 1. However, geotechnical engineers are commonly much better versed in deterministic methods than in probabilistic methods, and are generally more experienced and comfortable designing a structure to be safe with some appropriate conservatism than when making numerical assessments of the condition of existing and perhaps marginal structures. To provide some initial methodology for the latter problem, the approach of this study is to:

a. Review the performance modes of concern to existing levees loaded by floods and the related deterministic models for assessing performance.

b. Review the use of probabilistic methods in geotechnical engineering, hydraulic structures, and related areas.

c. Recommend procedures for developing reliability curves or conditional probability of failure functions similar to Figure 1 that are sufficiently simple for use in practice with limited data and a modest level of effort, but reflect a geotechnical engineer's understanding of the underlying mechanics and uncertainty in the governing parameters.

d. Test and illustrate the procedures through two comprehensive example problems.

2 Current Corps of Engineers' Guidance

In this chapter, current Corps of Engineers' guidance regarding levee planning and design is reviewed in order to begin to define the component parts of, and the constraints on, a probabilistic procedure to evaluate existing levees. One policy letter has been issued which defines a beginning point for these studies:

> Policy Guidance Letter No. 26, *Benefit Determination Involving Existing Levees* (23 Dec. 1991).

A second document, Engineer Manual (EM) 1110-2-1913, *Design and Construction of Levees* (U.S. Army Corps of Engineers 1978), is the primary source of Corps policy on the engineering aspects of levee design. However, probabilistic methods are not considered in this engineering manual. In addition to the EM, there exists a voluminous collection of research reports, flood performance reports, and Division regulations, (all developed by the Corps), as well as journal papers and reference books, that deal with the analysis and design of levees.

Policy Guidance Letter No. 26, Benefit Determination Involving Existing Levees (23 Dec 1991)

This letter sets forth the need (of the planner to receive from the engineer) for a function relating levee reliability to floodwater elevation, or at least two points on this function. Several specific items in the letter are especially relevant to the present study. These are quoted below and followed by a commentary.

> **Quote:** *Investigations ... involving the evaluation ...of existing levees and the related effect on the economic analysis shall use a systematic approach to resolving indeterminate, or arguable, degrees of reliability.*
>
> **Comment:** This language sets forth the requirement for applying the principles of reliability analysis to the problem.

Quote: *Studies ...will focus on the sources of uncertainty ... surface erosion, internal erosion (piping), underseepage, and slides...*

Comment: This wording summarizes the most commonly expected modes of adverse performance prior to overtopping. These will be considered in the developed methods.

Quote: *The question to be answered is: what percent of the time will a given levee withstand water at height x?*

Comment: This wording provides the specific requirement for developing the conditional probability of failure function defined in Chapter 1.

Quote: *...commands...(i.e. Corps district and division offices) making reliability determinations should gather information to enable them to identify two points... The highest vertical elevation on the levee such that it is highly likely that the levee would not fail if the water surface would reach this level... shall be referred to as the Probable Non-Failure Point (PNP)... The lowest vertical elevation on the levee such that it is highly likely that the levee would fail... shall be referred to as the Probable Failure Point (PFP).. As used here, "highly likely" means 85+ percent confidence...*

Comment: The definition of two specific points, the PNP and the PFP, implies the assumption of linearity noted later in the letter. The defined levels of reliability (0.85 / 0.15 and 0.15 / 0.85) assigned to these points, along with illustrated definitions (Figure 2a), permit an economist, in the absence of any further engineering analysis, to quantify reliability as a linear function based on two points derived from engineering analysis or engineers' intuition and judgment. The engineer needs only to, by some means, identify floodwater elevations for which he or she considers the levee to be 15 and 85 percent reliable.

Quote: *The requirement that as the water surface height increases the probability of failure increases, incorporates the reasonable assumption that as the levee is more and more stressed, it is more and more likely to fail.*

Comment: While this would often be the case, it should be noted that there may be some cases, notably riverside slope stability, where a levee may be more reliable or safe when loaded with floodwater than before or after flooding.

Quote: *If the form of the probability distribution is not known, a **linear relationship** as shown in the enclosed example, is an acceptable approach for calculating the benefits associated with the existing levees.*

> **Comment:** The assumption of linearity is certainly expedient, and is the least-biased assumption in a case where two and only two points on a function are known and no other information is present. However, the assumption of linearity may or may not be acceptable once some additional information *is* known. One of the objectives of this research is to determine what is in fact a reasonable function shape based on the results of some engineering analyses for typical levee cross sections and typical parameter values.

The attachment to the Policy Guidance Letter provides an illustration of the assumed linear conditional probability of failure function. In Figures 2a, 2b, and 2c, respectively, of this report are sketched the linear version, a trilinear version that could be extended from the linear version, and the general curves from Figure 1. The latter have been redrawn to show Pr_f, the dependent variable, on the y-axis. In the Policy Guidance Letter, the shape of the curve below the 0.15 value and above the 0.85 value is not defined; the tri-linear version shown is merely a representation of one possible interpretation. It will be seen from the results of the example analyses that the conditional-probability-of-failure functions generally take the shape of the middle curve in Figure 2c and can be approximated by a piecewise linear approach using three or more pieces similar to Figure 2b.

EM 1110-2-1913, "Design and Construction of Levees"

The current primary source of levee design guidance in the Corps of Engineers is EM 1110-2-1913, *Design and Construction of Levees* (U.S. Army Corps of Engineers 1978). Guidance in EM 1110-2-1913 relevant to the reliability assessment of existing levees includes the following:

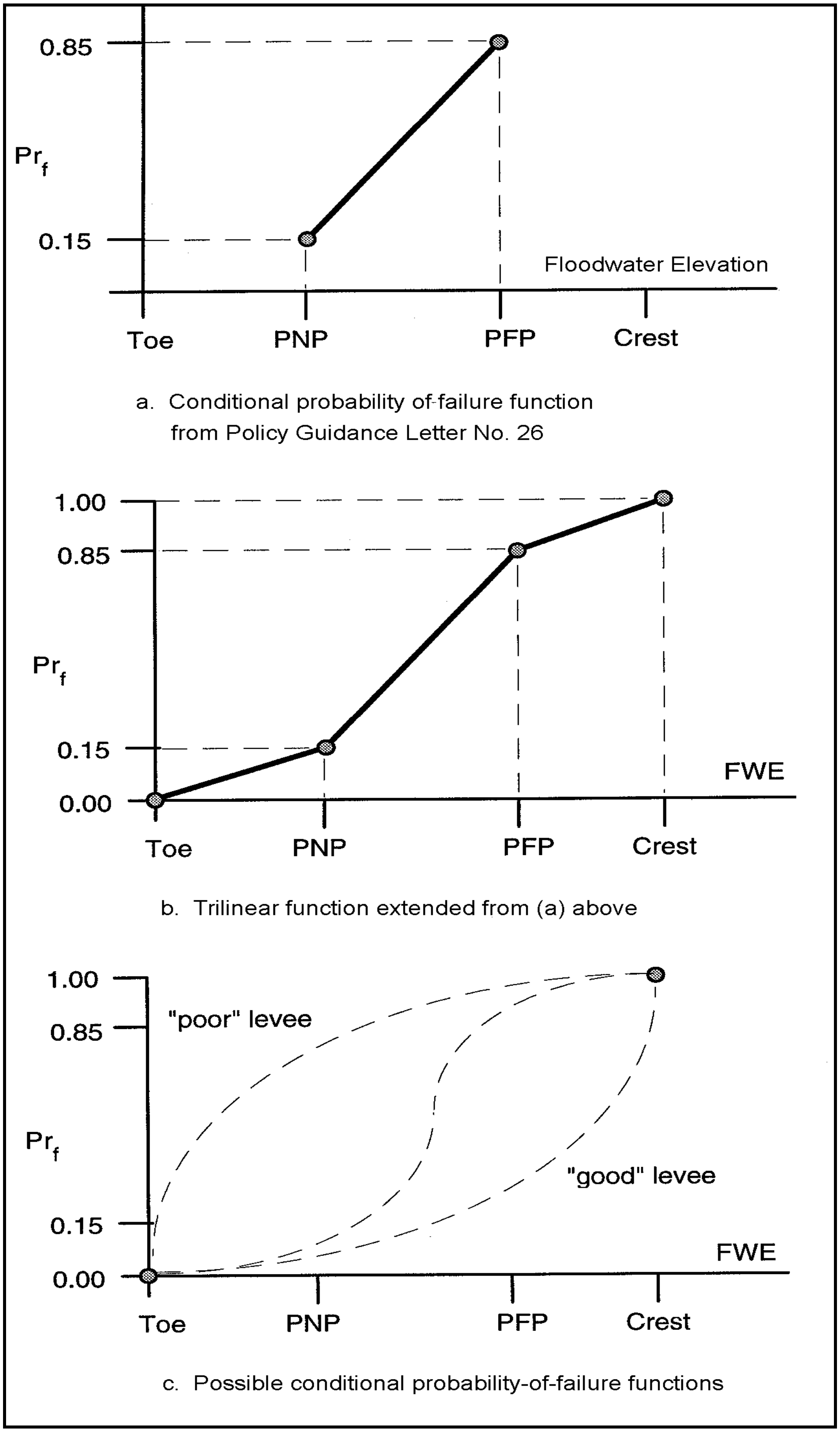
a. Q tests (UU tests) are recommended for determining the **strength of foundation clays**.

b. Q, R, and S tests (UU, CU, and CD tests) are recommended for determining **strength of borrow materials** compacted to water contents and densities consistent with expected field compaction.

c. For familiar foundation conditions, **undrained strength of fine-grained soil** may be estimated from consolidation stresses and Atterberg limits ($c/p = f(PI)$) and drained strength may be estimated from Atterberg limits data ($\phi' = f(PI)$) .

d. **Strength of pervious soils** is estimated from S (CD) tests on similar soils or correlations such as those given by NAVFAC DM-7.

e. **Permeability of pervious soils** is estimated from grain size information, specifically D_{10} size.



a. Conditional probability of-failure function from Policy Guidance Letter No. 26

b. Trilinear function extended from (a) above

c. Possible conditional probability-of-failure functions

Figure 2. Alternative conditional probability-of-failure functions

f. **Berms** of 40-ft width (riverside) to 100-ft width [1] (landside) are recommended to be left at natural ground elevation between the levee and borrow areas.

g. At least 2 ft of **impervious cover** should be left over pervious materials in borrow areas.

h. Although **underseepage control** is discussed, no criteria are given. The reader is referred to TM 3-424 (U.S. Army Corps of Engineers 1956).

i. **Through-seepage** and defensive works such as toe drains and internal drains are described; however, no design criteria are presented and it is noted that provision of such defenses is usually uneconomical. Underseepage and through-seepage for dams are discussed in EM 1110-2-1901, "Seepage Analysis and Control for Dams" (U.S. Army Corps of Engineers 1986). A design procedure for toe berms to provide stability against through-seepage for sand levees has been developed by Schwartz (1976) and the Rock Island District.[1]

j. A 1V:2.5H slope is considered the steepest that can be **maintained with mowing equipment.**

k. **Freeboard** (crest height above design flood) is recommended to be at least 2 ft in agricultural areas and 3 ft in urban areas, with additional height in critical areas.

l. **Crown width** is recommended to be a minimum width of 10 to 12 ft for floodfighting operations.

m. **Slope stability analyses** may be in accordance with the Modified Swedish Method or the wedge method from EM 1110-2-1902, or the simpler Swedish Slide Method (ordinary method of slices). It would be expected that current practice may also be to use Spencer's method from computer programs UTEXAS2 or UTEXAS3 and not to use the simpler Swedish Slide Method. In the EM, five stability cases are identified; of these, Case I (end-of-construction) and Case V (earthquake) are not considered of interest for economic assessment of existing levees; the remaining cases (sudden drawdown, intermediate river stage, and steady seepage) are to be considered.

n. **Embankment construction deficiencies** leading to poor performance are summarized in Table 7-2 of the EM. Relevant items include organic material not stripped from the foundation, highly organic fill, excessively

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[1] A table of factors for converting non-SI units of measurement to SI units is presented on page B-11.

[1] Personal Communication, 1993, S. Zaidi, U.S. Army Engineer District, Rock Island; Rock Island, IL.

wet or dry fill, pervious layers through the embankment, and inadequate compaction.

o. **Erosion protection for riverside slopes** is discussed in general terms, but no quantitative criteria are given except where riprap is to be used, where another EM is referenced.

Components of an Improved Probabilistic Assessment Procedure

The current guidance for assessing the reliability of existing levees essentially consists of the following:

a. Using the template method and/or slope stability analysis to determine stable slopes that meet accepted criteria.

b. Defining the PNP and PFP from these slope stability considerations

c. Adjusting the PNP and PFP, if necessary, by some judgmental means, based on the sum total of information gleaned from the field inspection.

It is proposed that a more rational and consistent assessment procedure should include the following components.

a. Develop a set of **conditional probability-of-failure versus floodwater elevation functions,** one for each of the following performance considerations:

(1) **Underseepage** using established Corps methods (closed-form equations or numerical methods such as program LEVEEMSU) and engineering reliability analysis. Geometry may be based on field surveys, minimal borings, and geologic experience; permeability values may be based on correlations with grain size and experience.

(2) **Slope stability for short-term conditions,** where undrained strengths related to consolidation stresses are used for impervious materials and drained strengths for pervious materials, using a slope stability program and engineering reliability analysis. Strengths may be based on field data where available or on correlations and experience for preliminary studies.

(3) **Slope stability for long-term conditions,** where flood duration is expected to be sufficiently long that pore pressures adjust to flood conditions, using drained strengths, infinite slope analysis or slope stability programs, and engineering reliability analysis. Strengths may be based on correlations and experience.

(4) **Through-seepage** leading to internal erosion (piping) or surface erosion of the landside slope. For sand levees, several methods are considered in Chapter 9; these can likely be further refined based on additional studies. Results may be modified based on engineering judgment and observations from the field inspection regarding materials, geometry, vegetation in the levee, crown width, likelihood of animal burrows, cracks, roots, defects, etc.

(5) **Surface erosion** due to current and wave attack on the riverside slope, using engineering judgment and observations from the field inspection regarding soil cover, vegetative cover, river characteristics, wave exposure, etc. As techniques are further developed, these analyses can be based on probabilistic definitions of current velocities, wave properties, and the properties of levee cover materials.

b. **Systematically combine these functions** into one composite conditional-probability-of failure function for a given floodwater elevation, using accepted methods from probability theory.

c. Using the results of steps a and b for a few selected levee reaches, **incorporate length effects** to estimate the conditional-probability-of-failure function for the entire levee system.

Such a scheme will be developed and illustrated in Chapters 4 through 11. Before doing so, related research work by others will be briefly reviewed in Chapter 3.

3 Related Research

Before developing the procedures and examples herein, a brief review of the engineering literature on levees, their primary modes of performance (i.e., slope stability, seepage, etc.), and the application of probabilistic methods thereto was made to provide a basis for model development and to take advantage, if possible, of previous work in the field. This section summarizes recent (within the past 20 years) work relevant to the topic. It is not intended to be a comprehensive review of levee engineering. In making the review, it became clear that nearly all work on levees and flood control embankments published in English derives from the experiences of three sources: the Corps of Engineers in the United States, Dutch engineers involved in sea dike construction, and Czech engineers involved in protection from flooding along the Danube.

Comprehensive Works

Peter (1982), in *Canal and River Levees*, provides the most complete and recent reference book treatise on levee design, based on work in the former Czechoslovakia. Notable among Peter's work is a more up-to-date and extended treatment of mathematical and numerical modeling than in most other references. (His numerical treatment of the underseepage problem was part of the inspiration for the numerical approach used in LEVEEMSU.) Peter also considers under-seepage safety as a function of particle size and size distribution, and not just gradient alone. Although Peter's work was not directly used in this study, it bears consideration and re-review as the probabilistic approach to levee assessment is further extended and developed by the Corps of Engineers.

Vrouwenvelder (1987), in *Probabilistic Design of Flood Defenses*, provides a very thorough treatise on a probabilistic approach to the design of dikes and levees in the Netherlands. At this time, the report does not have the status of a code, but reviews the status of research activities and provides worked examples illustrating how dike design can be cast as a risk management problem. High-lights of Vrouwenvelder's work potentially relevant to this effort include the following:

a. It is recognized that **exceedance frequency of the crest elevation is not taken as the frequency of failure**; there is some probability of failure for lower elevations, and there is some probability of no failure or inundation above this level if an effort is made to raise the protection.

b. A problem-specific **review of probabilistic concepts** such as event trees, fault trees, reliability analysis (limit state, performance function, etc.), and series and parallel systems is provided.

c. In his example, **eleven parameters are taken as random variables** which are used in conjunction with relatively simple mathematical physical models.

d. **Performance modes** considered are overflowing and overtopping, macro-instability (deep sliding), micro-instability (shallow sliding or erosion of the landside slope due to seepage), and piping (as used, equivalent to underseepage as termed by the Corps).

e. Aside from overtopping, piping (**underseepage**) **is found to be the governing mode** for the section studied; slope stability is of little significance to probability of failure.

f. **Surface erosion** due to wave attack or parallel currents is **not considered**.

g. For analysis of **macro-instability** (deep sliding), the Bishop method is used, and previous data from Alonso (1976) is cited that indicates pore pressure and cohesion dominate the uncertainty. This is consistent with findings of this writer in the study of Corps' dams (Wolff 1985, 1989).

h. For analysis of **micro-instability** (shallow landside sloughing), a limit equilibrium derivation, essentially equivalent to the "infinite slope" method of EM 1110-2-1902 (U.S. Army Corps of Engineers 1970) is used.

i. For analysis of piping (**underseepage**), the Lane and Bligh creep ratio approaches were originally used and then supplanted by an empirical model test procedure that incorporates the D_{50} size and coefficient of uniformity of the foundation sands. Research is under way toward the development of a grain-transport model and the consideration of time-dependent effects.

j. The **"length problem"** (longer dikes are less reliable than equivalent short ones) is discussed.

k. An **example** probabilistic design is provided for a 20-km-long river dike constructed of sand with a cover of clay. Random variables include:

Water height and duration
Soil permeability *k*
Soil friction angle
Soil cohesion *c*'
Equivalent permeability of the (top blanket) clay $k_{k.eq}$
Equivalent thickness of the (top blanket) clay $d_{k.eq}$
Equivalent leakage factor of the clay facing $\lambda_{eq} = k_{k.eq} / d_{k.eq}$
Model uncertainty factor for piping, based on Lane's creep ratio

l. The **probabilistic procedure is aimed at optimizing the height and slope angle of new dikes** with respect to total costs for construction and expected losses, including property and life. Macro-instability (slope failure) of the inner slope was found to have a low risk, much less than 8×10^{-4} per year. Piping was found to be sensitive to seepage path length; probabilities of failure varied but were several orders of magnitude higher (10^{-2} to 10^{-3} per year). Micro-instability (landside sloughing due to seepage) was found to have very low probabilities of failure. Based on these results, it was determined that only overtopping and piping need be considered in the combined reliability evaluation.

Slope Stability

Termaat and Calle (1994) describe studies made to evaluate the short-term acceptable risk of slope failure of levees being reconstructed along rivers in Holland. Using a slope stability analysis procedure (Calle 1985) that considers a random field model of spatial fluctuation of shear strength combined with a Bishop type slope stability model cast in a second-moment probabilistic analysis, the factor of safety is determined as a Gaussian random function in the direction of the length of the levee. The expected value, standard deviation, and auto correlation function for the factor of safety are determined by the random field statistics of the shear strength functions. From these, estimates of the probability of occurrence of a zone where the factor of safety is below 1.0 somewhere along the slope axis can be obtained along with an indication of the width of such a zone. The authors conclude that probabilities of failure for the end-of-construction condition are on the order of 1 in 200, which is consistent with the findings of a number of other researchers. Although the spatial correlation considerations used by Termaat, Calle and others are beyond the scope of this preliminary study of levee reliability, these are important factors that should be considered as the methodology is further developed.

Underseepage, Through-Seepage, and Piping

Calle et al. (1989), all with Delft Geotechnics in The Netherlands, developed a probabilistic procedure for analyzing the likelihood of piping beneath sea dikes and river levees. Whereas Corps models for underseepage (U.S. Army Corps of Engineers 1956) are based on considerations of equilibrium necessary to initiate a sand boil, Calle's model considers the dynamic equilibrium necessary to accelerate or terminate erosion and material movement once piping has initiated. The latter phenomenon is related to the creep ratio, originally defined by Bligh (1910) and Lane (1935). The critical creep ratio defines a limit state which explicitly depends on geometrical and physical parameters of the aquifer and its sand material. These parameters, which are modeled as seven random variables and one deterministic variable, include the D_{10} and D_{70} grain sizes, the permeability, the length of the structure, and the soil friction angle. Using the Hasofer-Lind (1974) reliability formulation, the reliability index can be calculated for a levee and foundation system under consideration. This in turn is used to calculate the partial factors of safety on the creep ratio necessary to make the probability of piping small relative to the annual risk of overtopping (1 in 12,500 for the Dutch structures considered). In doing so, it was found that creep ratios on the order of two-thirds those recommended by Bligh would provide adequate reliability against uncontrolled movement of material.

Multiple Modes of Failure

Duckstein and Bogardi (1981) applied reliability theory to levee design, considering the combined effects of overtopping, boiling, slope sliding, and wind wave erosion. However, specific models for geotechnical aspects such as boiling or slope sliding are not developed in detail. Instead, each performance mode i is characterized by a critical height H_i for which failure would occur, and the H_i values are taken as a set of random variables. The combined probability is obtained as a union of the conditional probabilities, similar in concept to the scheme used in Chapter 11 of this report.

Duncan and Houston (1981) summarize studies performed for the Sacramento District to estimate failure probabilities for California levees constructed of a heterogeneous mixture of sand, silt, and peat, and founded on peat of uncertain strength. Stability failure was analyzed using a horizontal sliding block model driven by the riverside water load. The factor of safety is expressed as a function of the shear strength, which is a random variable due to its uncertainty, and the water level, for which there is a defined annual exceedance probability. Using elementary probability theory, values for the annual probability of failure for 18 islands in the levee system were calculated by numerically integrating over the joint events of high water levels and insufficient shear strength. At this point, the obtained probability of failure values were adjusted based on several practical considerations; first, they were normalized with respect to length of levee reach modeled (longer reaches should be more likely to have a failure) and secondly, they were adjusted from relative probability values to more absolute values by adjusting them with respect to the observed number of failures. These practical concepts are of significance to many or most ongoing developments in applying probabilistic procedures to practical problems by the Corps of Engineers.

4 Two Example Problems Defined

In this chapter, hypothetical levee cross sections for two example problems are defined. These are considered to represent two points along a broad range of levee problems that may be encountered by an engineer in practice. In subsequent chapters, these two sections will be used to illustrate analyses for slope stability, seepage, and erosion. The examples involve:

a. A sand levee with a thin topsoil facing on a thin uniform clay top stratum.

b. A clay levee on a thick, nonuniform clay top stratum.

For each example section, the semipervious clay top stratum is assumed to be underlain by a thick pervious substratum.

Problem 1: Sand Levee on Thin Uniform Clay Top Stratum

Example problem 1 consists of a 20-ft-high sand levee with 1V:2.5H side slopes and a 20-ft-wide crown. It is founded on an 8-ft-thick clay top blanket which is in turn underlain by an 80-ft-thick pervious sand substratum. The crown width of 10 ft is between the 8-ft and 12-ft values corresponding to the PFP and PNP templates. The 1V:2.5 slopes are steeper than recommended for either template and represent a slope at the margin of maintainability. A levee section for example problem 1 is shown in Figure 3.

Problem 2: Clay Levee on Thick NonUniform Clay Top Stratum

Example problem 2 consists of a 20-ft-high clay levee with 1V:2H side slopes and a 10-ft-wide crown. It is founded on a semipervious clay top blanket which is 20 ft thick on the riverside of the levee. On the landside, the clay thickness increases to 30 ft at the levee toe where a plugged channel parallels the levee.

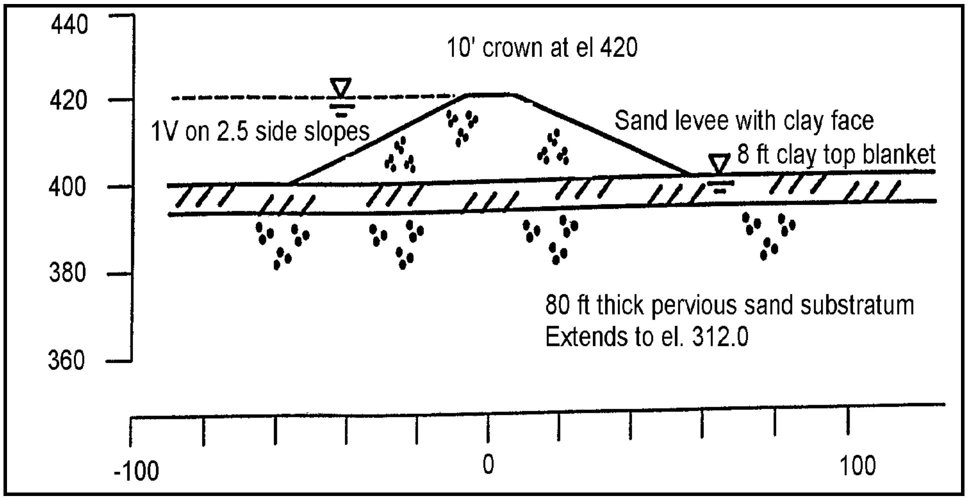

Figure 3. Levee section for example problem 1

Landside of the levee toe, the ground elevation drops 5 ft in 40 ft and the clay blanket thins to 15 ft, creating a location for a potential seepage concentration 80 ft landside of the levee center line. The top stratum is underlain by a pervious sand substratum extending to elevation 312.0. Figure 4 is a levee section for example problem 2.

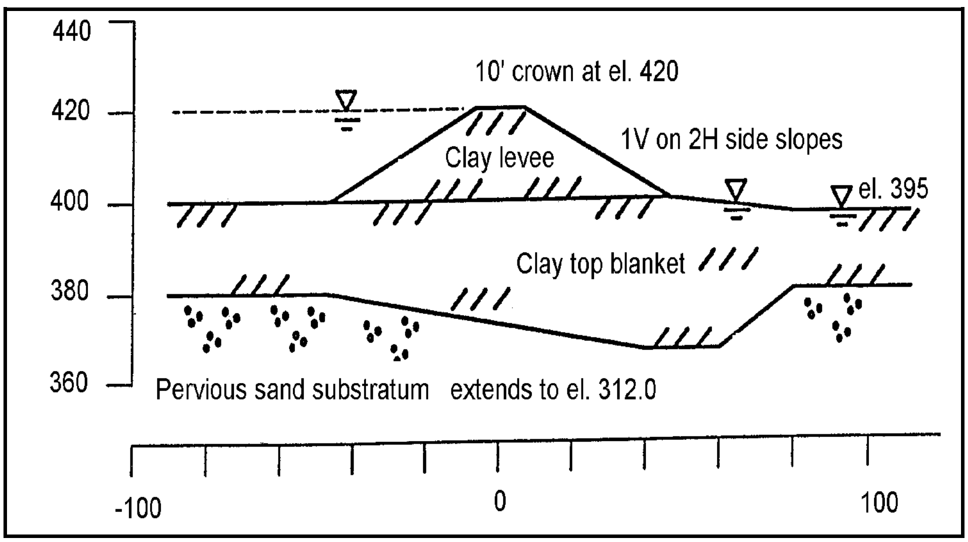

Figure 4. Levee section for example problem 2

The crown width of 10 ft corresponds to the PNP template (PFP is 6 ft) and the 1V:2H side slopes correspond to the PFP template and the margin of maintainability.

5 Characterizing Uncertainty in Geotechnical Parameters

Introduction

The *capacity-demand model*, described in Annex A and used herein to calculate probabilities of failure, requires that the engineer assign values for the probabilistic moments of the random variables considered in analysis. This chapter reviews information regarding the observed variability of geotechnical parameters and can be used as a guide when characterizing random variables for the analysis of levees.

Any parameter used in a geotechnical analysis can be modeled as a random variable, and any variables that are expected to contribute uncertainty regarding the expected performance of the structure or system should be so modeled. Typically these include soil strength and soil permeability. In the Taylor's Series first-order second moment (FOSM) approach used herein, random variables are quantified by their expected values, standard deviations, and correlation coefficients, commonly referred to as probabilistic *moments*. These moments are defined in Annex A. Depending on the quantity and quality of available information, values for probabilistic moments may be estimated in one of several ways:

a. From statistical analysis of test data measuring the desired parameter.

b. From index test data which may be correlated to the desired parameter.

c. Simply based on judgment and experience where test data are not available.

Each step from the top to the bottom in the above list implies increasing uncertainty. When designing a new structure, the move from using test data to using index data or from using index data to using experience only would likely be accompanied by an increase in the factor of safety or an adjustment in the value of a design parameter (e.g. reducing the design strength). The corresponding action in reliability analysis would be to assume a larger coefficient of variation.

Table 1 provides a summary of typical reported values for the coefficients of variation of commonly encountered geotechnical parameters. More detailed comment regarding the observed variability of relevant parameters is provided in the subsequent sections.

Table 1
Coefficients of Variation for Geotechnical Parameters

Parameter	Coefficient of Variation, percent	Reference
Unit weight	3 4 to 8	Hammitt (1966), cited by Harr (1987) assumed by Shannon and Wilson, Inc., and Wolff (1994)
Drained strength of sand ϕ'	3.7 to 9.3 12	Direct shear tests, Mississippi River Lock and Dam No. 2, Shannon and Wilson, Inc., and Wolff (1994) Schultze (1972), cited by Harr (1987)
Drained strength of clay ϕ'	7.5 to 10.1	S tests on compacted clay at Cannon Dam, Wolff (1985)
Undrained strength of clay s_u	40 30 to 40 11 to 45	Fredlund and Dahlman (1972) cited by Harr (1987) Assumed by Shannon and Wilson, Inc., and Wolff (1994) Q tests on compacted clay at Cannon Dam, Wolff (1985)
Strength-to-effective stress ratio s_u / σ'_v	31	Clay at Mississippi River Lock and Dam No. 2, Shannon and Wilson, Inc., and Wolff (1994)
Coefficient of permeability k	90	For saturated soils, Nielson, Biggar, and Erh (1973) cited by Harr (1987)
Permeability of top blanket clay k_b	20 to 30	Derived from assumed distribution, Shannon and Wilson, Inc., and Wolff (1994)
Permeability of foundation sands k_f	20 to 30	For average permeability over thickness of aquifer, Shannon and Wilson, Inc., and Wolff (1994)
Permeability ratio k_f / k_b	40	Derived using 30% for k_f and k_b; see Annex B
Permeability of embankment sand	30	Assumed by Shannon and Wilson, Inc., and Wolff (1994)

Unit Weight of Soil Materials

The coefficient of variation of the unit weight of soil material is usually on the order of 3 to 8 percent. In slope stability problems, uncertainty in unit weight usually contributes little to the overall uncertainty, which is dominated by soil strength. For stability problems, it can usually be taken as a deterministic variable in order to reduce the number of random variables and simplify calculations. It may, however, require consideration for underseepage problems, where the critical exit gradient is directly proportional to the unit weight.

Drained Strength of Sands

Reported coefficients of variation for the friction angle (ϕ) of sands are in the range of 3 to 12 percent. Lower values can be used where there is some confidence that the materials considered are of consistent quality and relative density, and the higher values should be used where there is considerable uncertainty regarding material type or density. For the direct shear tests on sands from Lock and Dam No. 2 cited in Table 1 (Shannon and Wilson, Inc., and Wolff 1994), the lower coefficients of variation correspond to higher confining stresses and vice-versa.

Drained Strength of Clays

As the drained strength (ϕ') of clays is essentially a physical phenomenon similar to the drained strength for sands, similar coefficients of variation (3 to 12 percent) would be expected. Evaluation of S test data on compacted clays at Cannon Dam (Wolff 1985) showed coefficients of variation in the range of 7.5 to 10 percent.

A common method in practice to estimate drained strength is by correlation to the plasticity index. Correlations developed by the Corps of Engineers are shown in the engineering manual on design and construction of levees (U.S. Army Corps of Engineers 1978). Holtz and Kovacs (1981) summarize correlations developed by Kenney (1959), Bjerrum and Simons (1960) and Ladd et al. (1977). Using such correlations, the observed variation in plasticity index for a clay deposit can be combined with the observed data scatter of the correlations in order to estimate coefficients of variation for drained strength parameters.

Undrained Strength of Clays

Estimation from test results

Where undrained tests are available on soils considered to be “representative” of a considered project area, the expected value and standard deviation of the undrained strength, s_u or c, may be estimated directly from statistical analysis of test data. An example is given in Table 2, which illustrates a statistical analysis of unconfined compression test data furnished by the St. Louis District. The resulting mean value and standard deviation of c, 1,234 and 798 lb/ft^2, respectively, might be rounded to the following estimated moments:

Expected value:	$E[c]$	=	1,200 lb/ft^2
Standard deviation:	σ_c	=	800 lb/ft^2
Coefficient of variation:	V_c	=	66.7 percent

Note, however, that the calculated coefficient of variation is very large, even larger than typical values cited in Table 1. In the case considered, samples were taken from a range of depths from about 2 to 20 ft, and hence had been consolidated under different effective overburden stresses. Where reasonable estimates of consolidation stress can be made, the uncertainty can be reduced if the undrained strength is normalized with respect to effective overburden stress as described in the next section. However, for the St. Louis data, even a regression analysis of strength versus sample depth did not reveal any trend. This suggests a "mixed population" of samples from different soil formations. Smaller coefficients of variation might be obtained if the soil samples can be separated into different strata based on visual examination, index property tests, and an understanding of the surficial geology.

Estimation from test results and consolidation stress

Ladd et al. (1977) and others have shown that the undrained strength s_u (or c) of clays with a given geologic origin can be "normalized" with respect to overburden stress (σ'_v) and overconsolidation ratio (OCR) and defined in terms of the ratio s_u/σ'_v. Analysis of test data on clay under the overflow dike for Mississippi River Lock and Dam No. 2 (Shannon and Wilson, Inc., and Wolff 1994) showed that it was reasonable to characterize uncertainty in clay strength in terms of the probabilistic moments of the s_u/σ'_v parameter. The ratio of s_u/σ'_v for 24 tested samples was found to have a mean value of 0.35, a standard deviation of 0.11, and a coefficient of variation of 31 percent.

Permeability for Seepage Analysis

Permeability of foundation sands

Permeability of sand samples can vary quite considerably; coefficients of variation of more than 100 percent have been reported. These large values are apparently the result of analyzing the variability of sand permeability from sample to sample. However, in an underseepage analysis, the variable of interest is not the permeability at the location of a specific sample, but the average permeability over the vertical extent of an aquifer at a selected cross section. For levee underseepage investigations, it is common to perform grain size analyses and obtain values for the D_{10} sizes at a number of points in a single boring. If these are used to estimate a set of permeability values using standard correlations (e.g., U.S. Army Corps of Engineers 1956), the expected value of the average permeability over the depth of the aquifer at the boring site can be taken as the mean value of the permeability estimates. The uncertainty in the average permeability over the section is smaller than the uncertainty in the permeability at a random point, and can be expressed as the standard error of the mean, which is the

Table 2
Example Statistical Analysis of Undrained Tests on Clay, Unconfined Compression Tests On Undisturbed Samples

Boring/Sample	W_{test}	W_{test} - W_{avg}	$(W_{test} - W_{avg})^2$	c	$c\text{-}c_{bar}$	$(c\text{-}c_{bar})^2$
VPC2-01-91U T-1	24.8	2.07	4.277376	750	-484.0909	234,344
VPCS-01-91U T-6	23.3	0.57	0.322831	1,600	365.9091	133,889.5
VPCS-01-91U T-7B	25.5	2.77	7.662831	1,350	115.9091	13,434.92
VPCS-02-91U T-2	20.5	-2.23	4.981012	750	-484.0909	234,344
VPCS-02-91U T-3	21.2	-1.53	2.346467	1,800	565.9091	320,253.1
VPCS-02-91U T-4	20.5	-2.23	4.981012	650	-584.0909	341,162.2
VPCS-02-91U T-6	20.4	-2.33	5.437376	650	-584.0909	341,162.2
VPCS-03-91U T-3B	24.1	1.37	1.871921	2,500	1,265.909	1,602,526
VPCS-03-91U T-4B	20.5	-2.23	4.981012	2,250	1,015.909	1,032,071
VPCS-03-91U T-5	21.9	-0.83	0.691921	2,850	1,615.909	2,611,162
VPPS-02-91U T-1	19.7	-3.03	9.191921	2,750	1,515.909	2,297,980
VPPS-02-91U T3	18.5	-4.23	17.90829	800	-434.0909	188,434.9
VPGD-01-91U ST-2	19.5	-3.23	10.44465	1,350	115.9091	13,434.92
VPGD-05091U ST-1	23.9	1.17	1.364649	900	-334.0909	111,616.7
VPL-10-91U S-1	19.5	-3.23	10.44465	1,350	115.9091	13,434.92
VPL-19-91U ST-2	21.5	-1.23	1.517376	500	-734.0909	538,889.5
VPL-19-91U ST-3	23.3	0.57	0.322831	400	-834.0909	695,707.6
VPL-19-91U ST-5	31.1	8.37	70.02647	250	-984.0909	968,434.9
VPL-22-91U S-1	17.6	-5.13	26.33556	2,100	865.9091	749,798.6
VPL-22-91U S-3	23.8	1.07	1.141012	350	-884.0909	781,616.7
VPL-22-91U S-5	27.4	4.67	21.79192	450	-784.0909	614,798.6
VPL-22-91U S-7	31.6	8.87	78.64465	800	-434.0909	188,434.9
Sum =	500.1		286.6877	27,150		14,026,932
N =	22		22	22		22
W_{avg} =	22.73	Var =	13.03126	C_{bar} = 1,234.0	Var =	637587.8
		Std. Dev. =	3.610		Std. Dev. =	798.491
		N - 1 =	21		N - 1 =	21
		Var =	13.6518		Var =	667949.1
		Std. Dev. =	3.695		Std. Dev. =	817.282

standard deviation of the sample values divided by the square root of the number of samples:

$$\sigma_{\bar{k}} = \frac{\sigma_k}{\sqrt{n}} \tag{3}$$

From detailed analysis of a number of borings near Lock and Dam No. 25 on the Mississippi River, the author (Shannon and Wilson, Inc., and Wolff 1994) measured coefficients of variation for the average sand permeability on the order of 20 to 30 percent.

Permeability of top blanket clays

Although intact clays may have coefficients of permeability in the range 10^{-6} to 10^{-9} cm/sec, values used to model the global permeability of a semipervious top stratum (k_b) are typically much larger, commonly on the order of 10^{-4} cm/sec, to reflect the effects of seepage through surface cracks, animal holes, and other defects. As the appropriate values have traditionally been estimated semi-empirically, using numbers back-calculated from observations during floods, typical values of the coefficient of variation are not accurately known. For studies of dikes along the Mississippi River, a coefficient of variation of 20 percent was assumed (Shannon and Wilson, Inc., and Wolff 1994), based on judgmental evaluation of the shape of trial probability distributions. For the underseepage studies in Chapter 6, a coefficient of variation of 30 percent was assumed for the top blanket.

Permeability ratio

The residual head landside of a levee and hence the potential for piping or boiling is in fact related to the ratio of the permeability of the pervious substratum to the permeability of the top blanket, k_f/k_b, and not to the absolute value of either permeability. If the expected values and standard deviations of the two parameters are known, the expected value and permeability of the ratio can be found as shown by example in Annex B.

6 Underseepage Analysis

In this chapter, levee underseepage analyses are illustrated for the two example problems defined in Chapter 4. The maximum exit gradient landside of the levee is taken as the performance function, and the value of the critical gradient, assumed to be 0.85, is taken as the limit state. As example problem 1 involves uniform foundation geometry, the classical methods of underseepage analysis given in TM3-424 (U.S. Army Corps of Engineers 1956a) are used to calculate the exit gradient at the levee toe. For example problem 2, which has an irregular foundation, the program LEVEEMSU (Wolff 1989) is used to calculate the maximum value of the exit gradient along a cross section perpendicular to the levee. Piezometric head profiles from these analyses are in turn used in the slope stability analyses of the next chapter.

Example Problem 1: Sand Levee on Thin Uniform Clay Top Stratum

The levee cross section for example problem 1 was illustrated in Figure 3. Four random variables are considered, the horizontal permeability of the pervious substratum k_f, the vertical permeability of the semi-pervious top blanket k_b, the thickness of the top blanket z, and the thickness of the pervious substratum d. The assigned probabilistic moments for these variables are given in Table 3.

Table 3
Random Variables for Example Problem 1

Parameter	Expected Value	Standard Deviation	Coefficient of Variation, Percent
Substratum permeability, k_f	1000×10^{-4} cm/sec	300×10^{-4} cm/sec	30
Top blanket permeability, k_b	1×10^{-4} cm/sec	0.3×10^{-4} cm /sec	30
Blanket thickness, z	8.0 ft	2.0 ft	25
Substratum thickness, d	80 ft	5 ft	6.25

The coefficients of variation of the top blanket and foundation permeability values (each 30 percent) were assigned based on the typical values summarized in Chapter 5.

As borings are not available at every possible cross section, there is some uncertainty regarding the thicknesses of the soil strata at the critical location. Hence, d and z are modeled as random variables. Their deviations are set to match engineering judgment regarding the probable range of actual values. For the blanket thickness z, assigning the standard deviation at 2.0 ft models a high probability that the actual blanket thickness will be between 4.0 and 12.0 ft ($\pm$ 2 standard deviations) and a very high probability that the blanket thickness will be between 2.0 and 14.0 ft ($\pm$ 3 standard deviations). For the aquifer thickness d, the two-standard-deviation range is 70 to 90 ft and the three-standard-deviation range is 65 to 95 ft. For analysis of real levee systems, it is suggested that the engineer review the geologic history and stratigraphy of the area and assign a range of likely strata thicknesses that are considered the thickest and thinnest probable values. These can then be taken to correspond to ± 2.5 to 3.0 standard deviations from the expected value.

As it is known that the exit gradient and stability against underseepage problems are functions of the permeability *ratio* k_f/k_b and not the absolute magnitude of the values, the number of calculations required for analyses can be reduced by treating the permeability ratio as a single random variable. To do so, it is necessary to determine the coefficient of variation of the permeability ratio given the coefficient of variation of the two permeability values. In Annex B of this report, example calculations are provided for three methods of calculating the moments of functions of random variables: the Taylor's series method with both exact and approximate derivatives, and the point estimate method. Based on these three examples, it appears reasonable to take the expected value of the permeability ratio as 1,000 and its coefficient of variation as 40 percent. This corresponds to a standard deviation of 400 for k_f/k_b.

To facilitate calculations, a spreadsheet (shown in Figure 5) was developed that accomplishes the following:

a. Solves for the exit gradient using the methods in TM3-424 (U.S. Army Corps of Engineers 1956a).

b. Repeats the solution for seven combinations of the input parameters required in the Taylor's series method.

c. Determines the expected value and standard deviation of the exit gradient.

d. Calculates the expected value and standard deviation of the natural logarithm of the exit gradient.

e. Calculates the probability that the exit gradient is above a critical value.

Underseepage Analysis
Levee on Infinite length foundation

T. F. Wolff
September 1994

H = 20
x2 = 110

	kf/kb	z	d	x3	s	ho	i	Variance component	% of variance
mean	1000	8	80	800.00	910.00	9.357	1.170		
	600	8	80	619.68	729.68	9.185	1.148		
	1400	8	80	946.57	1056.57	9.451	1.181	0.000276532	0.30
	1000	6	80	692.82	802.82	9.265	1.544		
	1000	10	80	894.43	1004.43	9.421	0.942	0.090606378	99.69
	1000	8	75	774.60	884.60	9.337	1.167		
	1000	8	85	824.62	934.62	9.375	1.172	5.55296E-06	0.01
							Total	0.090888464	100.00

E[i] = 1.170
Var[i] = 0.090888
sigma[i] = 0.301477
V(i) = 25.78%

E[ln i] = 0.12449
sigma[ln i] = 0.253629

i crit = 0.85

ln(i crit) = -0.16252

Pr(f) = 0.871101

Figure 5. Spreadsheet for underseepage analysis of example problem 2

Table 4
Problem 1, Underseepage Taylor's Series Analysis Water at Elevation 420 (H = 20 ft)

Run	k_f/k_b	z	d	h_o	i	Variance	Percent of Total Variance
1	1,000	8.0	80.0	9.357	**1.170**		
2	600	8.0	80.0	9.185	1.148	0.000276	0.30
3	1,400	8.0	80.0	9.451	1.181		
4	1,000	6.0	80.0	9.265	1.544	0.090606	99.69
5	1,000	10.0	80.0	9.421	0.942		
6	1,000	8.0	75.0	9.337	1.167	0.000006	0.01
7	1,000	8.0	85.0	9.375	1.172		
					Total	0.090888	100.0

Results from the spreadsheet for a 20-ft total head on the levee are summarized in Table 4. The details of the calculations follow.

For the first analysis (Run 1), the three random variables are all taken at their expected values. From TM3-424, first the effective exit distance x_3 is calculated as:

$$x_3 = \sqrt{\frac{k_f}{k_b} \cdot z \cdot d} = \sqrt{1000 \cdot 8 \cdot 80} = 800 \ ft \qquad (4)$$

As the problem is symmetrical, the distance from the riverside toes to the effective source of seepage entrance x_1 is also 800 ft.

From the geometry of the given problem, the base width of the levee x_2 is 110 ft.

The distance from the landside toe to the effective source of seepage entrance is:

$$s = x_1 + x_2 = 800 + 110 = 910 \ ft \qquad (5)$$

The net residual head at the levee toe is:

$$h_0 = \frac{Hx_3}{s + x_3} = \frac{20 \cdot 800}{910 + 800} = 9.357 \ ft \qquad (6)$$

And the landside toe exit gradient is:

$$i = \frac{h_0}{z} = \frac{9.357}{8.0} = 1.170 \tag{7}$$

For the second and third analyses, the permeability ratio is adjusted to the expected value plus and minus one standard deviation while the other two variables are held at their expected values. These are used to determine the component of the total variance related to the permeability ratio:

$$\left(\frac{\partial i}{\partial (k_f/k_b)}\right) \sigma^2_{(k_f/k_b)} \approx \left(\frac{i_+ - i_-}{2\sigma_{k_f/k_b}}\right)^2 \sigma^2_{k_f k_b} =$$

$$\left(\frac{i_+ - i_-}{2}\right)^2 = \left(\frac{1.181 - 1.148}{2}\right)^2 = 0.000277 \tag{8}$$

A similar calculation is performed to determine the variance components contributed by the other random variables.

When the variance components are summed, the total variance of the exit gradient is obtained as 0.090888. Taking the square root of the variance gives the standard deviation of 0.301.

The exit gradient is assumed to be a lognormally distributed random variable with probabilistic moments $E[i] = 1.170$ and $\sigma_i = 0.301$. Using the properties of the lognormal distribution described in Annex A, the equivalent normally distributed random variable has moments $E[ln\ i] = 0.124$ and $\sigma_{ln\ i} = 0.254$.

The critical exit gradient is assumed to be 0.85. The probability of failure is then:

$$Pr_f = Pr\ (\ln i > \ln 0.85) \tag{9}$$

This probability was evaluated using a normal distribution function built into the spreadsheet. It can be solved using standard tables by first calculating the standard normalized variate z:

$$z = \frac{1n\ i_{crit} - E[1n\ i]}{\sigma_{1ni}} = \frac{-0.16252 - 0.12449}{0.253629} = -1.132 \tag{10}$$

For this value, the cumulative distribution function $F(z)$ is 0.129, and represents the probability that the gradient is below critical. The probability that the gradient is above critical is

$$Pr_f = 1 - F(z) = 1 - 0.129 = 0.871 \tag{11}$$

Note that the z value is analogous to the reliability index β, and it could be stated that $\beta = -1.13$.

The probability calculation is illustrated in Figure 6. The exit gradient is taken to be lognormally distributed, making the natural log of the exit gradient normally distributed. The expected value of *ln i* (0.124) exceeds the limit state value (*ln i* = -0.163) by 0.287, or 1.132 standard deviations. The probability of having an exit gradient above critical is the area shaded. For a normal distribution, the probability of a value less than 1.132 standard deviations below the expected value or mean is 0.129; hence the probability of being above this point is 0.871.

Once the spreadsheet was complete, the analysis could be readily repeated for a range of heads on the levee from 0 to 20 ft. This was accomplished and the resulting conditional probability of failure function was plotted as shown in Figure 7. The shape of the function is similar to that suggested in Chapter 1. The probability of failure is very low until the head on the levee exceeds about 8 ft, after which it curves up sharply. It reverses curvature when heads are in the range 14 to 16 ft and the probability of failure is near 50 percent. When the floodwater elevation is near the top of the levee, the conditional probability of failure approaches 87 percent.

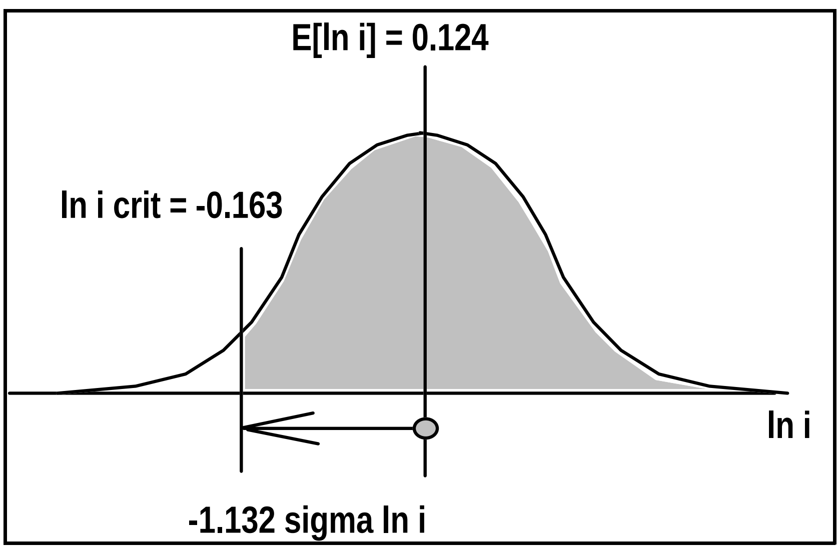

Figure 6. Calculation of probability of failure for underseepage

The results of one intermediate calculation in the analysis are worthy of note. As indicated by the relative size of the variance components shown in Table 4, virtually all of the uncertainty is in the top blanket thickness. A similar effect was found in other underseepage analyses by the writer reported in the Upper Mississippi River report (Shannon and Wilson, Inc., and Wolff 1994); where the

Example Problem 1

Conditional Probability of Underseepage Failure as a function of flood water height H

H	Pr(f)
0	0
2	9.99E-16
4	9.26E-08
6	1.50E-04
8	6.55E-03
10	5.47E-02
12	1.89E-01
14	3.92E-01
16	5.99E-01
18	7.63E-01
20	8.71E-01

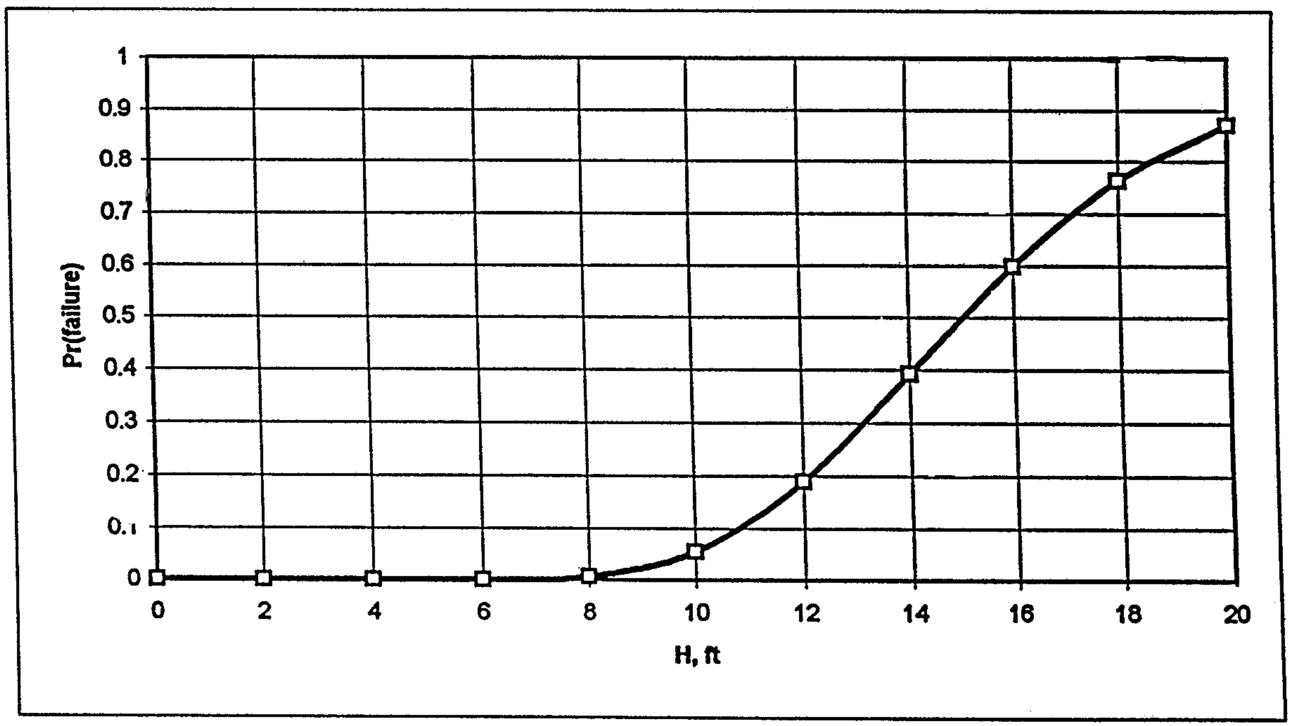

Figure 7. Conditional probability of failure function: Underseepage for example problem 1

top blanket thickness was treated as a random variable, its uncertainty dominated the problem. This has two implications:

a. Probability of failure functions for preliminary economic analysis might be developed using a single random variable, the top blanket thickness *z*.

b. In expending resources to design levees against underseepage failure, adding more data to the blanket thickness profile may be more justified than obtaining more data on material properties.

Example Problem 2: Clay Levee on Thick Non-Uniform Clay Top Stratum

Underseepage for example problem 2 was analyzed using the computer program LEVEEMSU (Wolff 1989), which is capable of analyzing irregular foundation geometry. Random variables were assigned the probabilistic moments shown in Table 5.

The permeability ratio k_f/k_b was modeled in LEVEEMSU by setting the top stratum permeability to 1 x 10^{-4} cm/sec and analyzing the foundation permeability at values of 1,000 x 10^{-4}, 600 x 10^{-4}, and 1400 x 10^{-4} cm/sec for the expected value, plus one standard deviation, and minus one standard deviation analyses, respectively.

Table 5
Random Variables for Example Problem 2

Parameter	Expected Value	Standard Deviation	Coefficient of Variation
Permeability ratio, k_f/kb	1,000	40	40%
Blanket thickness, z	As shown in Figure 6	2.0 ft	NA
Base of substratum elevation	312.0	5 ft	NA

Uncertainty in the blanket thickness was modeled by specifying the base of the blanket profile as shown in Figure 4 for the expected value and then moving it up and down 2 ft. This implies that the top blanket is assumed to be of the general shape shown and that there is a high probability that the blanket thickness is within ± 4 ft of the thickness shown and a very high probability that it is within ± 6 ft of the thickness shown.

Uncertainty in the base of the pervious substratum was likewise modeled by specifying it as shown and then moving it up and down 5 ft. This implies that there is a high probability that the base of the substratum is between elevation 302 and 322 (two standard deviations), and a very high probability that it is between elevations 297 and 327 (three standard deviations).

Results of the analyses for the maximum 20-ft head on the levee are as shown in Table 6.

A spreadsheet similar to that for problem 1 was developed to perform probability of failure calculations (Figure 8). For the maximum head of 20 ft on the levee, the expected value of the maximum exit gradient is 0.718 and its standard deviation is 0.0898. This corresponds to a probability of failure of 0.078, or almost 8 percent.

For lesser heads on the levee, it was assumed that the exit gradient is linear with respect to levee head, and the same spreadsheet was used with scaled exit gradient values (Figures 9 through 11) to calculate the probability of failure for lesser heads. At a 17.5-ft head, the probability of failure drops to 0.006, and at a 15-ft head, to 0.000097.

Table 6
Problem 2, Underseepage Taylor's Series Analysis Water at Elevation 420 (H = 20 ft)

Run	k_f/k_b	z	Base of Substratum	h_o at toe	I max	Variance	Percent of Total Variance
1	1000	E[z]	312.0		**.718**		
2	600	E[z]	312.0		.729	.000225	2.8
3	1400	E[z]	312.0		.699		
4	1000	+2.0	312.0		.640	.007832	97.1
5	1000	-2.0	312.0		.817		
6	1000	E[z]	317.0		.715	.000009	0.1
7	1000	E[z]	307.0		.721		
Total						.008066	100.0

The conditional probability of failure versus floodwater elevation is shown in Figure 12.

As was previously observed for example problem 1, examination of the variance terms indicates that virtually all of the uncertainty in the levee performance with respect to underseepage traces to uncertainty in the thickness of the top blanket: the thicker the top blanket or the more certain one is regarding the thickness of the blanket, the more reliable the levee can be considered.

Underseepage Analysis
Levee on Infinite length foundation

T. F. Wolff
September 1994

H = 20

	kf/kb	z	rock	i for H = 20	i		Variance component	% of variance
mean	1000	E[z]	312	0.718	0.718			
	1400	E[z]	312	0.729	0.729			
	600	E[z]	312	0.699	0.699		0.000225	2.79
	1000	+2.0	312	0.640	0.640			
	1000	-2.0	312	0.817	0.817		0.00783225	97.10
	1000	E[z]	317	0.715	0.715			
	1000	E[z]	307	0.721	0.721		9E-06	0.11
						Total	0.00806625	100.00

E[i] =	0.718	E[ln i] =	-0.33905
Var[i] =	0.008066		
sigma[i] =	0.089812	sigma[ln i] =	0.124602
V(i) =	12.51%		
i crit =	0.85	ln(i crit) =	-0.16252

Pr(f) = 0.078278

Figure 8. Spreadsheet for underseepage analysis of example problem 2 (*H* = 20 ft)

Underseepage Analysis
Levee on Infinite length foundation

T. F. Wolff
September 1994

H = 17.5

	kf/kb	z	rock	i for H = 20	i		Variance component	% of variance
mean	1000	E[z]	312	0.718	0.628			
	1400	E[z]	312	0.729	0.638			
	600	E[z]	312	0.699	0.612		0.000172266	2.79
	1000	+2.0	312	0.640	0.560			
	1000	-2.0	312	0.817	0.715		0.005996566	97.10
	1000	E[z]	317	0.715	0.626			
	1000	E[z]	307	0.721	0.631		6.89062E-06	0.11
						Total	0.006175723	100.00

E[i] =	0.628	E[ln i] =	-0.47258
Var[i] =	0.006176		
sigma[i] =	0.078586	sigma[ln i] =	0.124602
V(i) =	12.51%		
i crit =	0.85	ln(i crit) =	-0.16252

Pr(f) = 0.006416

Figure 9. Spreadsheet for underseepage analysis of example problem 2 (H = 17.5 ft)

Underseepage Analysis
Levee on Infinite length foundation

T. F. Wolff
September 1994

H = 15

	kf/kb	z	rock	i for H = 20	i		Variance component	% of variance
mean	1000	E[z]	312	0.718	0.539			
	1400	E[z]	312	0.729	0.547			
	600	E[z]	312	0.699	0.524		0.000126563	2.79
	1000	+2.0	312	0.640	0.480			
	1000	-2.0	312	0.817	0.613		0.004405641	97.10
	1000	E[z]	317	0.715	0.536			
	1000	E[z]	307	0.721	0.541		5.0625E-06	0.11
						Total	0.004537266	100.00

E[i] =	0.539	E[ln i] =	-0.62673
Var[i] =	0.004537		
sigma[i] =	0.067359	sigma[ln i] =	0.124602
V(i) =	12.51%		
i crit =	0.85	ln(i crit) =	-0.16252

Pr(f) = 0.000097

Figure 10. Spreadsheet for underseepage analysis of example problem 2 (*H* = 15 ft)

Underseepage Analysis
Levee on Infinite length foundation

T. F. Wolff
September 1994

H = 12.5

	kf/kb	z	rock	i for H = 20	i		Variance component	% of variance
mean	1000	E[z]	312	0.718	0.449			
	1400	E[z]	312	0.729	0.456			
	600	E[z]	312	0.699	0.437		8.78906E-05	2.79
	1000	+2.0	312	0.640	0.400			
	1000	-2.0	312	0.817	0.511		0.003059473	97.10
	1000	E[z]	317	0.715	0.447			
	1000	E[z]	307	0.721	0.451		3.51563E-06	0.11
						Total	0.003150879	100.00

E[i] =	0.449	E[ln i] =	-0.80905
Var[i] =	0.003151		
sigma[i] =	0.056133	sigma[ln i] =	0.124602
V(i) =	12.51%		
i crit =	0.85	ln(i crit) =	-0.16252

Pr(f) = 0.000000

Figure 11. Spreadsheet for underseepage analysis of example problem 2 (H = 12.5 ft)

Example Problem 2
Conditional Probability of Underseepage Failure as a function of flood water height H

H	Pr(f)
0.0	0
2.5	0
5.0	0
7.5	0
10.0	0
12.5	0
15.0	9.70E-05
17.5	6.42E-03
20.0	7.83E-02

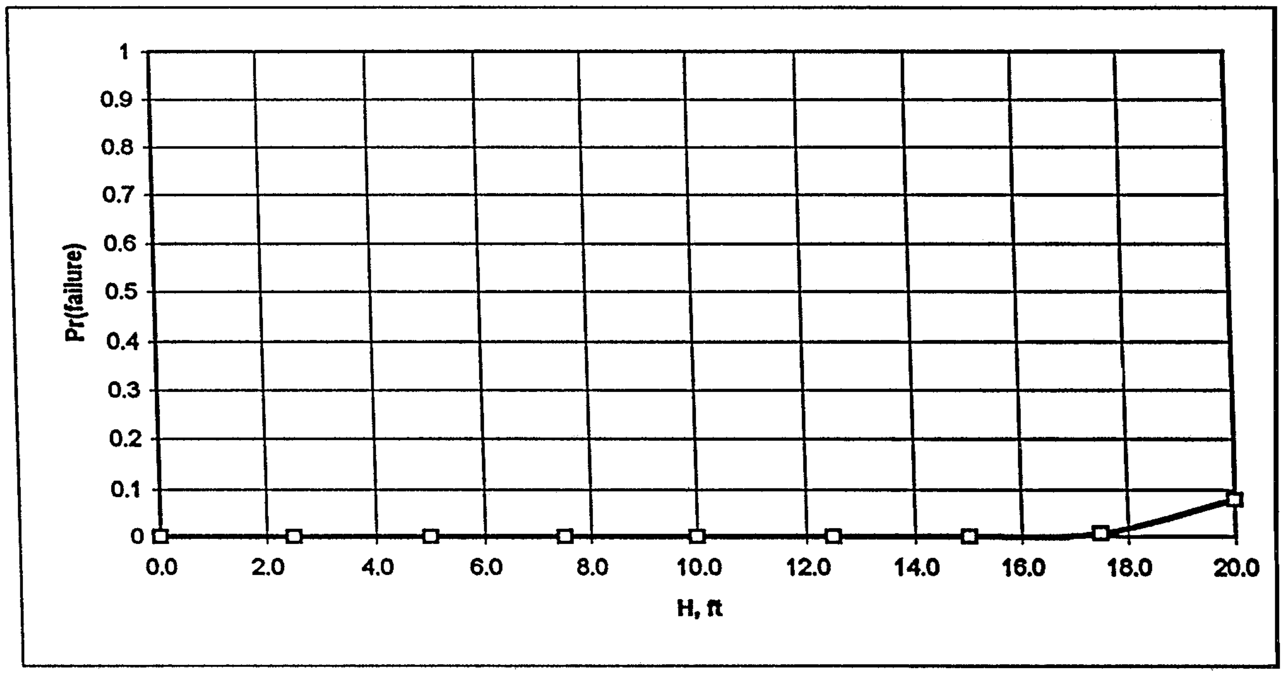

Figure 12. Conditional probability of failure function: Underseepage for example problem 2

7 Slope Stability Analysis for Short-Term Conditions

In this chapter, slope stability analyses are illustrated for the two example problems defined in Chapter 4 assuming undrained conditions prevail in the clay soils present in the profiles. This in turn implies that pore pressure conditions in the clay are dependent only on initial conditions prior to a flood and pore pressure changes due to shear, and that pore pressures have not equilibrated with flood water to develop steady-state seepage conditions in clay soils. These assumptions are consistent with short-term flood loadings. Slope stability analyses were performed using the computer program UTEXAS2 (Edris and Wright 1987). For the cases analyzed, similar results would be expected with the more recent program UTEXAS3.

Example Problem 1: Sand Levee on Thin Uniform Clay Top Stratum

Problem modeling

The levee cross section for example problem 1 was illustrated in Figure 3. For slope stability analysis, three random variables were defined; these variables, along with their assigned probabilistic moments, are summarized in Table 7.

Table 7
Random Variables for Example Problem 1

Parameter	Expected Value	Standard Deviation	Coefficient of Variation
Friction angle of sand levee embankment, ϕ_{emb}	30 deg	2 deg	6.7%
Undrained strength of clay foundation, c or s_u	800 lb/ft^2	320 lb/ft^2	40%
Friction angle of sand foundation, ϕ_{found}	34 deg	2 deg	5.9%

For slope stability analysis, the piezometric surface in the embankment sand was approximated as a straight line from the point where the floodwater intersects the riverside slope to the landside levee toe. For the internal erosion and through-seepage analyses in Chapter 9, this assumption is refined using Casagrande's basic parabola solution. The piezometric surface in the foundation sands was taken as that obtained for the expected value condition in the underseepage analysis reported in Chapter 6. If desired, the piezometric surface could be modeled as an additional random variable using the probabilistic moments of the residual head developed from the underseepage analysis.

Results

Using the Taylor's Series - Finite Difference method described in Annexes A and B, seven runs of the slope stability program are required for each floodwater level considered; one for the expected value case, and two runs to determine the variance component of each random variable. For the first water elevation considered (el. 400, or water at the natural ground surface), eleven runs were in fact made as several starting centers for the circular search option were checked to ensure that the critical failure surface was found. The results of the required seven runs are summarized in Table 8.

Table 8
Problem 1, Undrained Slope Stability, Taylor's Series Analysis Water at Elevation 400 (H = 0 ft)

Run	ϕ levee	c clay	ϕ found	FS	Variance	Percent of Total Variance
1-2	32	800	34	**1.568**		
4	30	800	34	1.448		
5	34	800	34	1.693	0.015 006	59.29
6	32	480	34	1.365		
7	32	1120	34	1.568	0.010 302	40.71
8	32	800	32	1.568		
9	32	800	36	1.567	2.5×10^{-7}	0.00
			Total		0.025 309	100.0

The results for all runs for all water elevations are summarized in Table 9. Critical failure surfaces for the cases of floodwater at elevation 400, 410, and 420 are illustrated in Figures 13 through 15. The reliability index and probability of failure for each water elevation were calculated using the spreadsheet templates illustrated in Figures 16 through 21. The resulting conditional probability of failure function is illustrated in Figure 22 and enlarged in Figure 23.

Table 9
Problem 1, Undrained Slope Stability, Results for All Runs

	Material Properties			Water	Initial Values			Final Critical Surface				Initial Values			Final Critical Surface			
Run #	φ(Emb)	c (clay)	φ(Fnd)	Elev	X	Y	Tang	FS	X	Y	Tang	X	Y	Tang	FS	X	Y	Tang
1A	32	800	34	400	50	450	400	1.568	63	444	400	50	450	390	2.009	31.6	432.6	392
4A	30	800	34	400	50	450	400	1.449	63	444	400	50	450	390	1.448	41.6	461.4	409.2
5A	34	800	34	400	50	450	400	1.693	63	444	400	50	450	390	2.042	31.6	432.4	392
6A	34	480	34	400	50	450	400	1.568	63	444	400	50	450	390	1.365	31.4	431.4	392
7A	32	1120	34	400	50	450	400	1.568	63	444	400	50	450	390	2.548	41.2	439.6	388
8A	32	800	32	400	50	450	400	1.568	63	444	400	50	450	390	2.150	41.2	441	389.6
11A	32	800	36	400	50	450	400	1.568	63	444	400	50	450	390	1.567	41.6	461.4	409.2
12A	32	800	34	410	50	450	400	1.568	63	444	400	50	450	390	1.961	43.4	442.2	388.4
13A	30	800	34	410	50	450	400	1.449	63	444	400	50	450	390	1.936	43.6	443	388.6
14A	34	800	34	410	50	450	400	1.449	63	444	400	50	450	390	1.987	43.2	441.8	388.4
15A	32	480	34	410	50	450	400	1.693	63	444	400	50	450	390	1.332	31.2	431.6	392
16A	32	1120	34	410	50	450	400	1.568	63	444	400	50	450	390	2.248	43	440.2	386.4
17A	32	800	32	410	50	450	400	1.568	63	444	400	50	450	390	1.897	43.2	441.4	388
18A	32	800	36	410	50	450	400	1.568	63	444	400	50	450	390	2.023	43.6	443.4	389
19A	32	800	34	405	50	450	400	1.568	63	444	400	50	450	390	2.105	42.4	442	389.2
20A	30	800	34	405	50	450	400	1.449	63	444	400	50	450	390	2.077	42.4	442.4	389.4
21A	34	800	34	405	50	450	400	1.693	63	444	400	50	450	390	2.134	42.2	441.4	389.2
22A	32	480	34	405	50	450	400	1.568	63	444	400	50	450	390	1.359	31.4	431.4	392
23A	32	1120	34	405	50	450	400	1.568	63	444	400	50	450	390	2.410	42	439.8	387.2
24A	32	800	32	405	50	450	400	1.568	63	444	400	50	450	390	2.038	42.2	440.8	388.6
25A	32	800	36	405	50	450	400	1.568	63	432.6	400	50	450	390	2.169	42.4	442.6	389.8

(Continued)

Table 9 (Concluded)

	Material Properties				Initial Values			Final Critical Surface				Initial Values			Final Critical Surface			
Run #	ϕ (Emb)	c (clay)	ϕ(Fnd)	Water Elev	X	Y	Tang	FS	X	Y	Tang	X	Y	Tang	FS	X	Y	Tang
26A	32	800	34	415	50	450	400	1.502	50	452	400	50	450	390	1.774	44.6	443.6	387.8
27A	30	800	34	415	50	450	400	1.387	50	452.8	400	50	450	390	1.753	44.6	444.4	388
28A	34	800	34	415	50	450	400	1.584	48.6	454	400	50	450	390	1.794	44.6	443.2	387.6
29A	32	480	34	415	50	450	400	1.502	50	452.8	400	50	450	390	1.420	45.8	445.8	390
30A	32	1120	34	415	50	450	400	1.502	50	452.8	400	50	450	390	2.049	44	441.6	386
31A	32	800	32	415	50	450	400	1.502	50	452.8	400	50	450	390	1.717	44.4	442.4	387.2
32A	32	800	36	415	50	450	400	1.502	50	452.8	400	50	450	390	1.829	45	444.8	388.2
33A	32	800	34	420	50	450	400	1.044	50	454	400	50	450	390	1.504	46.8	449.2	387.4
34A	30	800	34	420	50	450	400	0.995	50.8	452.8	400	50	450	390	1.490	47.0	449.4	387.4
35A	34	800	34	420	50	450	400	1.162	50.8	452.8	400	50	450	390	1.515	46.4	448.6	387.4
36A	32	480	34	420	50	450	400	1.044	50	454	400	50	450	390	1.158	48.8	454.8	389.4
37A	32	1120	34	420	50	450	400	1.044	50	454	400	50	450	390	1.775	45.4	445.6	385.8
38A	32	800	32	420	50	450	400	1.044	50	454	400	50	450	390	1.457	46.2	448	387
39A	32	800	36	420	50	450	400	1.044	50	454	400	50	450	390	1.546	47.2	450.4	387.8
50A	32	800	34	417.5	50	450	400	1.339	50.0	444.4	400.0	50	450	390	1.601	45.0	444.4	387.0
51A	30	800	34	417.5	50	450	400	1.237	50.0	444.4	400.0	50	450	390	1.636	45.6	445.8	387.6
52A	34	800	34	417.5	50	450	400	1.446	50.0	444.8	400.0	50	450	390	1.670	45.2	444.8	387.4
53A	32	480	34	417.5	50	450	400	1.339	50.0	444.4	400.0	50	450	390	1.307	47.0	449.4	389.6
54A	32	1120	34	417.5	50	450	400	1.339	50.0	444.4	400.0	50	450	390	1.926	44.6	442.8	385.8
55A	32	800	32	417.5	50	450	400	1.339	50.0	444.4	400.0	50	450	390	1.601	45.0	444.4	387.0
56A	32	800	36	417.5	50	450	400	1.339	50.0	444.4	400.0	50	450	390	1.703	45.8	446.6	388.0

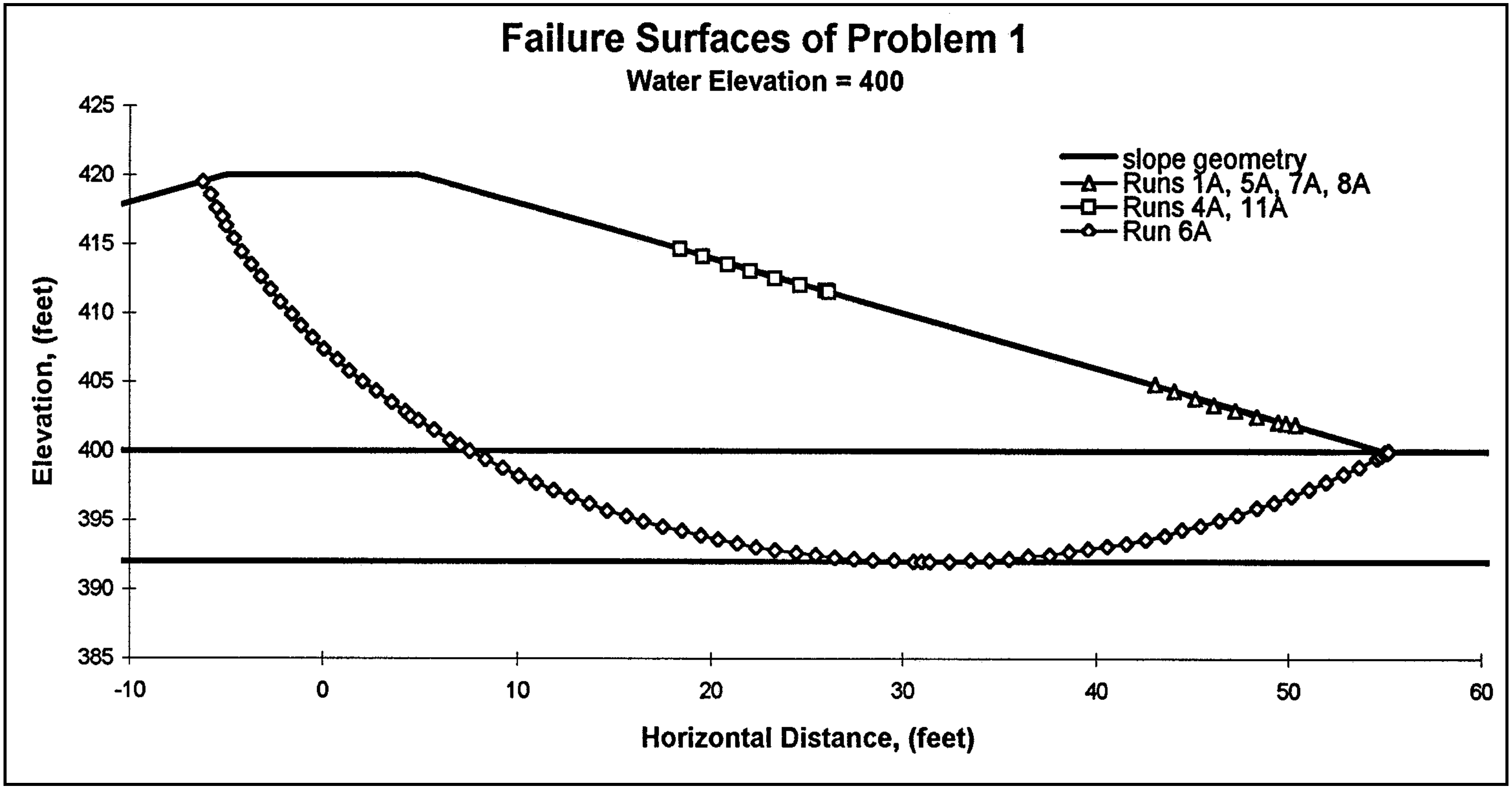

Figure 13. Failure surfaces for example problem 1, water elevation = 400

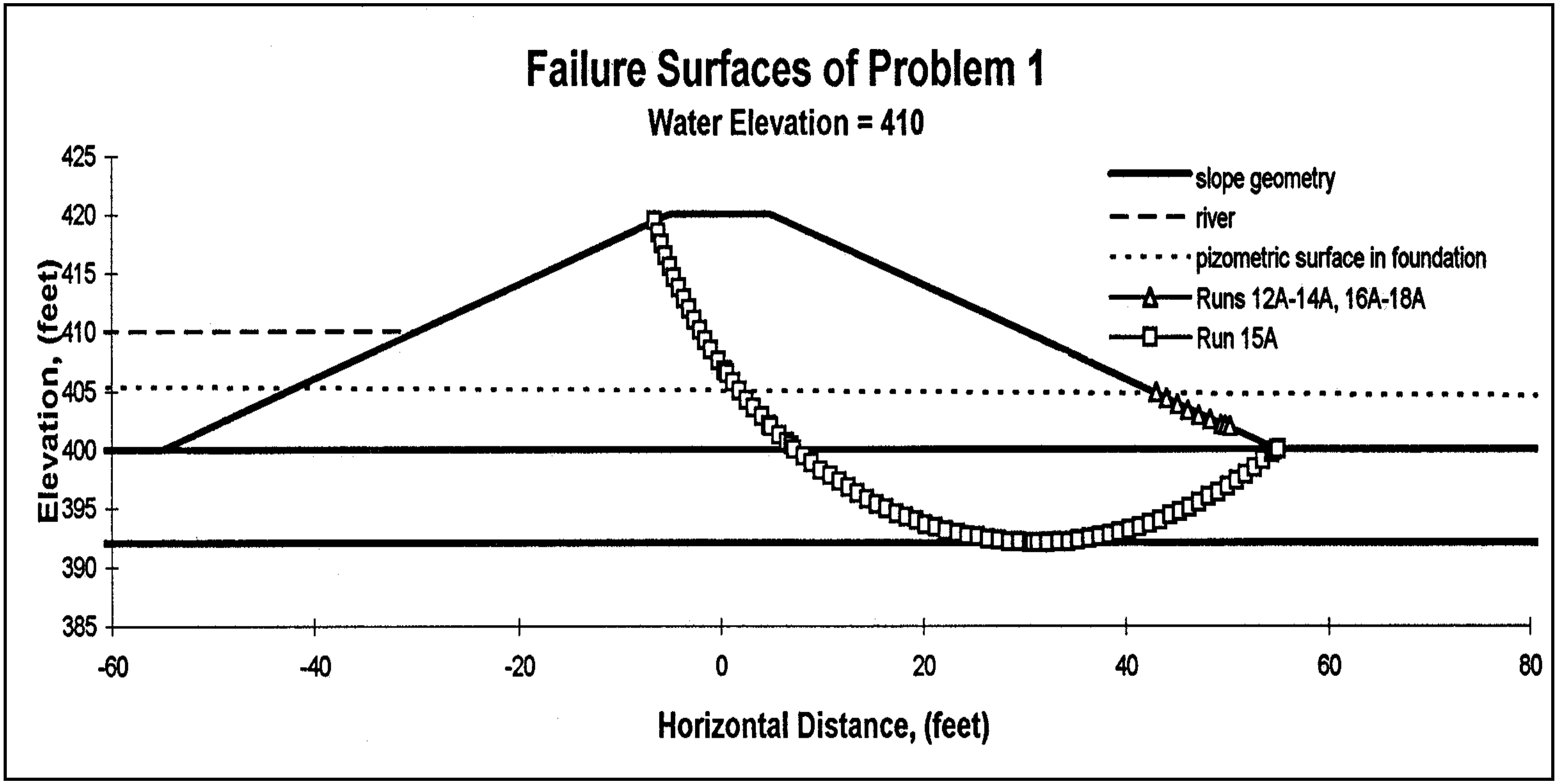

Figure 14. Failure surfaces for example problem 1, water elevation = 410

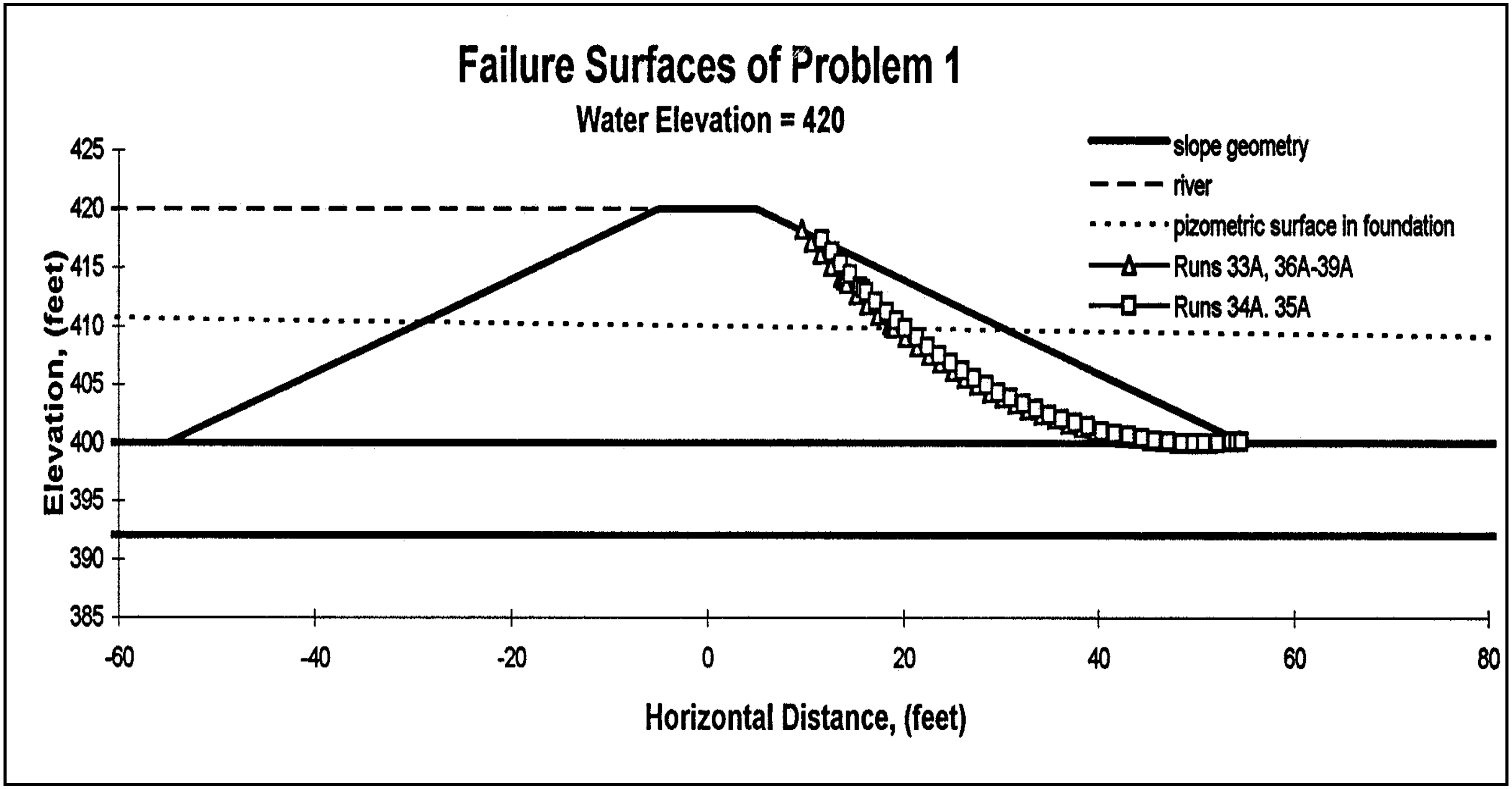

Figure 15. Failure surfaces for example problem 1, water elevation = 420

Slope Stability Analysis
Problem 1

Wolff / Ramon / Rahat
June 1995

Water Ht =	0
Water El =	400

	ϕ(emb)	c (clay)	ϕ(sub)	FS	Variance component	% of variance
mean	**32**	**800**	**34**	**1.568**		
	30	800	34	1.448		
	34	800	34	1.693	0.0150063	59.29
	32	480	34	1.365		
	32	1120	34	1.568	0.0103023	40.71
	32	800	32	1.568		
	32	800	36	1.567	2.5E-07	0.00
				Total	0.0253088	100.00

E[FS] =	**1.568**	E[ln FS] =	0.4446803
Var[FS] =	0.025309		
sigma[FS] =	**0.159087**	sigma[ln FS] =	0.101199
V(FS) =	10.15%		

Beta = **4.394115591**

FS crit =	1	ln(FS crit)	0

Pr(f) = **0.000006**

Figure 16. Reliability calculations for undrained slope stability, example problem 1, water height = 0, water elevation = 400

Slope Stability Analysis
Problem 1

Wolff / Ramon / Rahat
June 1995

Water Ht = 5
Water El = 405

	φ(emb)	c (clay)	φ(sub)	FS	Variance component	% of variance
mean	**32**	**800**	**34**	**1.568**		
	30	800	34	1.449		
	34	800	34	1.693	0.014884	57.68
	32	**480**	34	1.359		
	32	**1120**	34	1.568	0.0109203	42.32
	32	800	**32**	1.568		
	32	800	**36**	1.568	0	0.00
				Total	0.0258043	100.00

E[FS] = 1.568
Var[FS] = 0.025804
sigma[FS] = 0.160637
V(FS) = 10.24%

E[ln FS] = 0.4445806
sigma[ln FS] = 0.1021798

Beta = 4.35096396

FS crit = 1
ln(FS crit) 0

Pr(f) = 0.000007

Figure 17. Reliability calculations for undrained slope stability, example problem 1, water height = 5, water elevation = 405

Slope Stability Analysis
Problem 1

Wolff / Ramon / Rahat
June 1995

Water Ht = 10
Water El = 410

	φ(emb)	c (clay)	φ(sub)	FS	Variance component	% of variance
mean	**32**	**800**	**34**	**1.568**		
	30	800	34	1.449		
	34	800	34	1.693	0.014884	51.67
	32	**480**	34	1.332		
	32	**1120**	34	1.568	0.013924	48.33
	32	800	**32**	1.568		
	32	800	**36**	1.568	0	0.00
				Total	0.028808	100.00

E[FS] =	**1.568**	E[ln FS] =	0.4439764
Var[FS] =	0.028808		
sigma[FS] =	**0.169729**	sigma[ln FS] =	0.1079306
V(FS) =	10.82%		
FS crit =	1	ln(FS crit)	0

Beta = **4.11353701**

Pr(f) = 0.000019

Figure 18. Reliability calculations for undrained slope stability, example problem 1, water height = 10, water elevation = 410

Slope Stability Analysis
Problem 1

Wolff / Ramon / Rahat
June 1995

Water Ht = 15
Water El = 415

	φ(emb)	c (clay)	φ(sub)	FS	Variance component	% of variance
mean	**32**	**800**	**34**	**1.502**		
	30	800	34	1.387		
	34	800	34	1.584	0.0097023	85.23
	32	**480**	34	1.420		
	32	**1120**	34	1.502	0.001681	14.77
	32	800	**32**	1.502		
	32	800	**36**	1.502	0	0.00
				Total	0.0113833	100.00

E[FS] = 1.502
Var[FS] = 0.011383
sigma[FS] = 0.106692
V(FS) = 7.10%

E[ln FS] = 0.404281
sigma[ln FS] = 0.0709441

Beta = 5.6985824

FS crit = 1

ln(FS crit) 0

Pr(f) = 0.000000

Figure 19. Reliability calculations for undrained slope stability, example problem 1, water height = 15, water elevation = 415

Slope Stability Analysis
Problem 1

Wolff / Ramon / Rahat
June 1995

Water Ht =	17.5
Water El =	417.5

	ϕ(emb)	c (clay)	ϕ(sub)	FS	Variance component	% of variance
mean	**32**	**800**	**34**	**1.339**		
	30	800	34	1.237		
	34	800	34	1.446	0.0109203	97.71
	32	**480**	34	1.307		
	32	**1120**	34	1.339	0.000256	2.29
	32	800	**32**	1.339		
	32	800	**36**	1.339	0	0.00
				Total	0.0111763	100.00

E[FS] =	**1.339**	E[ln FS] =	0.288816
Var[FS] =	0.011176		
sigma[FS] =	**0.105718**	sigma[ln FS] =	0.0788302
V(FS) =	7.90%		
FS crit =	1	ln(FS crit)	0

Beta = **3.66377481**

Pr(f) = **0.000124**

Figure 20. Reliability calculations for undrained slope stability, example problem 1, water height = 17.5, water elevation = 417.5

Slope Stability Analysis
Problem 1

Wolff / Ramon / Rahat
June 1995

Water Ht =	20
Water El =	420

	φ(emb)	c (clay)	φ(sub)	FS	Variance component	% of variance
mean	**32**	**800**	**34**	**1.044**		
	30	800	34	0.995		
	34	800	34	1.162	0.0069722	100.00
	32	**480**	34	1.044		
	32	**1120**	34	1.044	0	0.00
	32	800	**32**	1.044		
	32	800	**36**	1.044	0	0.00
				Total	0.0069722	100.00

E[FS] =	**1.044**	E[ln FS] =	0.0398712
Var[FS] =	0.006972		
sigma[FS] =	**0.0835**	sigma[ln FS] =	0.0798534
V(FS) =	8.00%		
FS crit =	1	ln(FS crit)	0

Beta = **0.49930523**

Pr(f) = **0.308782**

Figure 21. Reliability calculations for undrained slope stability, example problem 1, water height = 20, water elevation = 420

Example Problem 1

Conditional probability of slope failure as a function of flood water height H

H	Pr(f)
0	0.000006
5	7.00E-06
10	1.90E-05
15	6.06E-09
17.5	0.000124
20	3.09E-01

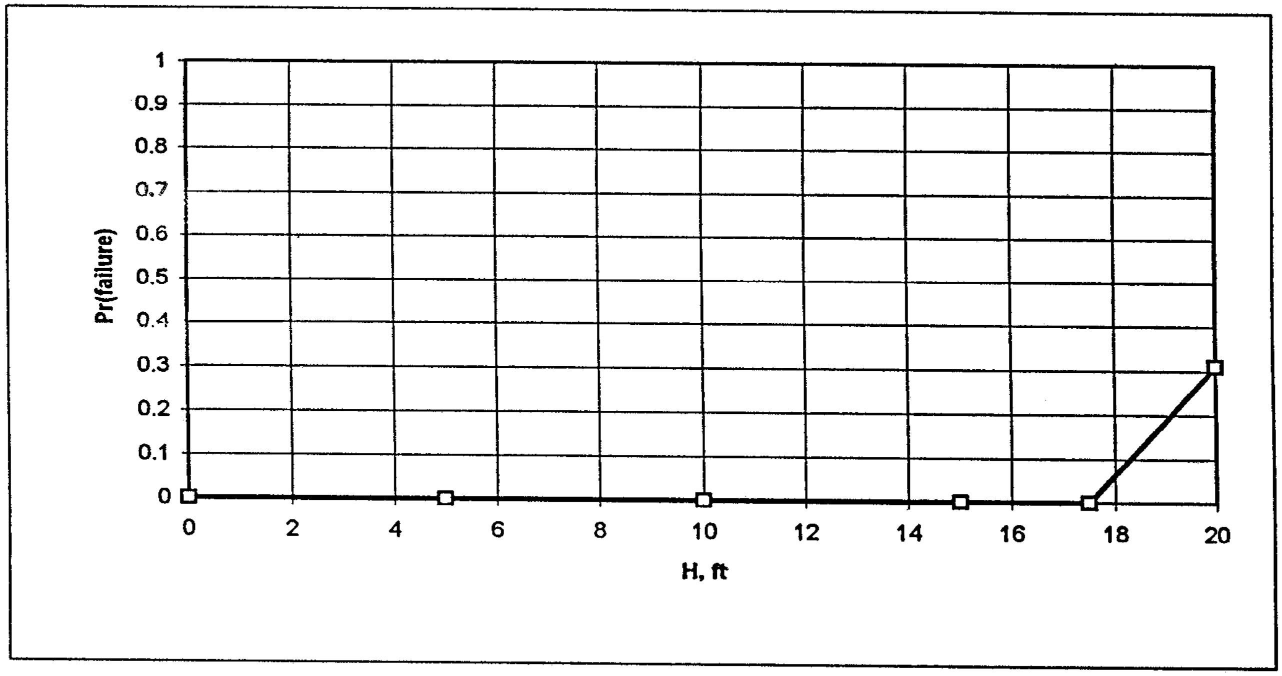

Figure 22. Conditional probability function for undrained slope failure, example problem 1

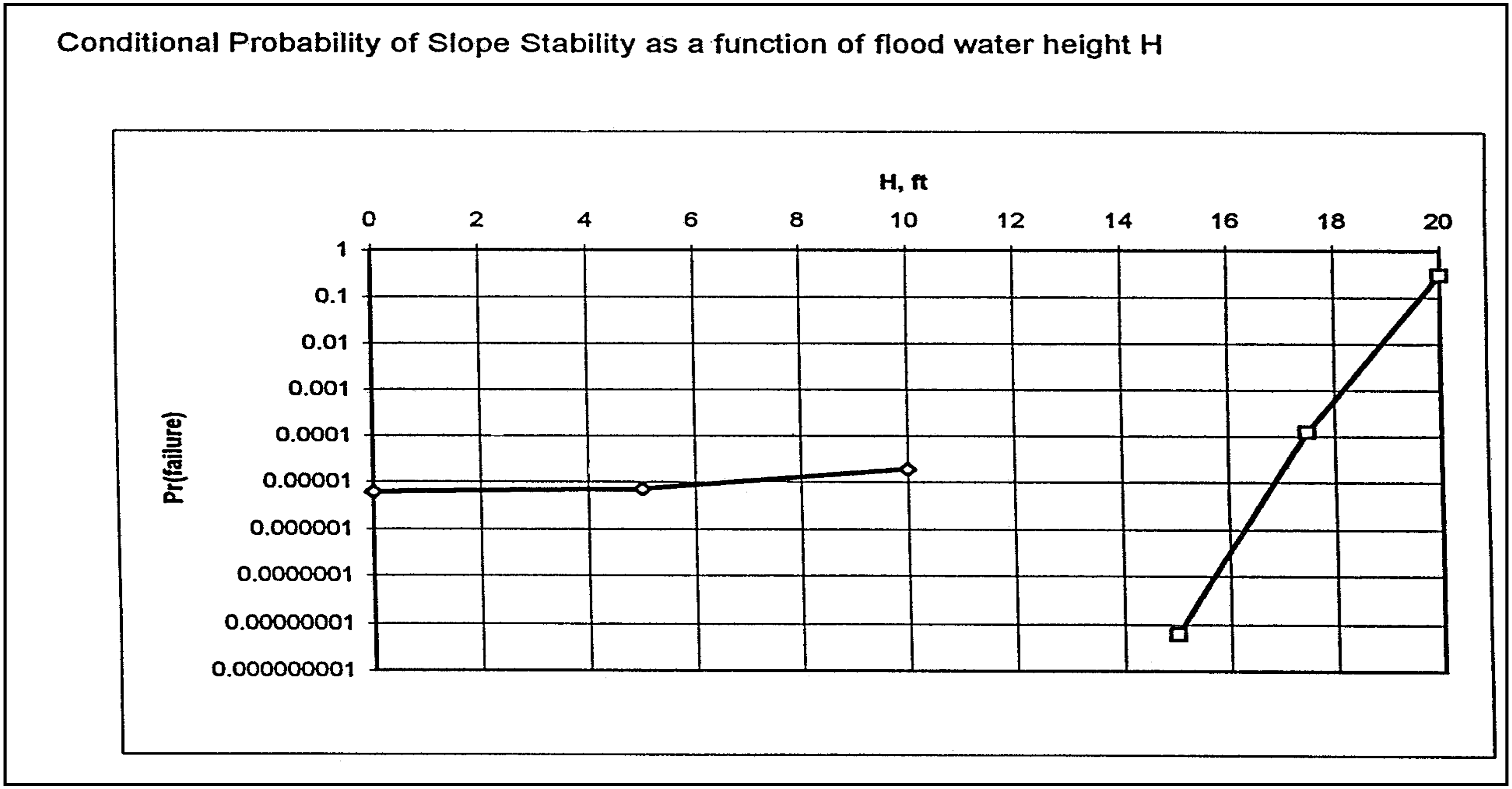

Figure 23. Conditional probability function for undrained slope failure, example problem 1, enlarged view

A discontinuity in Pr_f is observed as the flood height is increased from 10 ft to 15 ft; Pr_f abruptly decreases, then begins to rise again. This illustrates an interesting facet of probability analysis; Pr_f is a function not only of the expected values of the factor of safety and the underlying parameters, but also of their coefficients of variation. In the present case, at a flood height between 10 ft and 15 ft, some of the critical surfaces move from the foundation clay, with a high coefficient of variation for its strength, to the embankment sands, for which the coefficient of variation is smaller. This decreases β and Pr_f. Even though the safety factor may decrease as the flood height increases, if the value of the smaller safety factor is more certain, due to the lesser strength uncertainty, Pr_f may decrease.

Example calculation of probability values

The calculation of the probability values for the case of water at elevation 400 is summarized as follows.

The expected value of the factor of safety is the factor of safety calculated using the expected values of all variables:

$$E[FS] = 1.568 \tag{12}$$

The variance of the factor of safety, calculated in the same manner as previously illustrated for the exit gradient in underseepage in the previous chapter, is:

$$Var[FS] = 0.025309 \tag{13}$$

and the standard deviation of the factor of safety is:

$$\sigma_{FS} = 0.159 \tag{14}$$

While the factor of safety is expected to be adequate (1.568), its exact value is uncertain. The factor of safety is assumed to be a lognormally distributed random variable with $E[FS] = 1.568$ and $\sigma_{FS} = 0.159$. From the properties of the lognormal distribution given in Annex A,

$$V_{FS} = \frac{\sigma_{FS}}{E[FS]} = \frac{0.159}{21.568} = 0.1015 \tag{15}$$

$$\sigma_{InFS} = \sqrt{1n(1 + V_{FS}^2)} = \sqrt{1n(1 + 0.1015^2)} = 0.1012 \tag{16}$$

$$E[1nFS] = 1nE[FS] - \frac{\sigma^2_{1nFS}}{2} = 1n1.568 - \frac{0.0102}{2} = 0.4447 \qquad (17)$$

The reliability index is then:

$$\beta = \frac{E[1nFS]}{\sigma_{1nFS}} = \frac{0.447}{0.1012} = 4.394 \qquad (18)$$

From the cumulative distribution function of the standard normal distribution evaluated at -β, the conditional probability of failure for water at elevation 400 is:

$$Pr_f = 6 \ x \ 10^{-6} \qquad (19)$$

The calculation of the reliability index is illustrated in Figure 24.

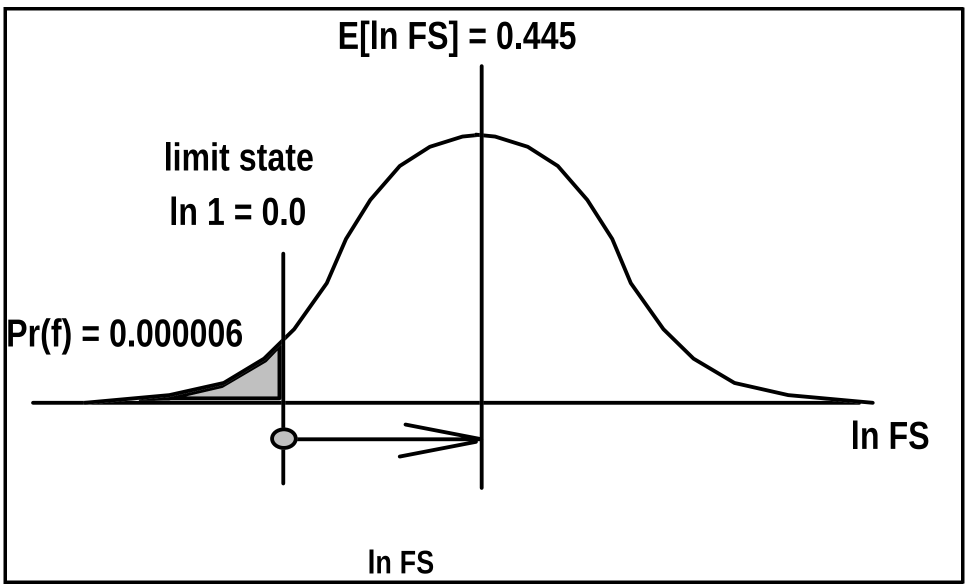

Figure 24. Calculation of probability of failure for slope stability

Interpretation

Note that the calculated probability of failure infers that the existing levee is taken to have approximately a six in one million probability of not being stable under the condition of floodwater to its base elevation of 400, even though it may in fact be existing and observed stable under such conditions. The capacity-demand / reliability index model was developed for the analysis of yet-unconstructed structures. When applied to existing structures, it will provide probabilities of failure greater than zero. This can be interpreted as follows: given a large number of different levees, each with the same geometry and with the variability in the strength of their soils distributed according to the same density functions as those assigned by the engineer to characterize uncertainty in the soil strength, about six in one million of those levees might be expected to have slope stability problems. The expression of reliability of existing structures in this manner provides a consistent probabilistic framework for use in economic evaluation of improvements to those structures.

Discussion

The results of the probabilistic analyses are summarized in Table 10.

Table 10
Problem 1, Slope Stability for Short-Term Conditions, Summary of Probabilistic Analyses

Water Elevation	E[FS]	σ_{FS}	β	Pr_f
400.0	1.568	0.159	4.394	6×10^{-6}
405.0	1.568	0.161	4.351	7×10^{-6}
410.0	1.568	0.170	4.114	1.9×10^{-5}
415.0	1.502	0.107	5.699	6×10^{-9}
420.0	1.044	0.084	0.499	0.3087

As would be expected, the anticipated value of the factor of safety decreases with increasing floodwater elevation. Contrary to what might be expected, the reliability index increases and the probability of failure decreases with increasing floodwater elevation until the floodwater exceeds elevation 415.0, or three quarters the levee height. This occurs because the uncertainty in the factor of safety decreases along with the expected value, and the probability of failure reflects both measures. Although the factor of safety becomes smaller as the floodwater rises, its value becomes more dependent on the shear strength of the embankment sands and less dependent on the shear strength of the foundation clays. This is evident in Figure 14 where the failure surface moves down into the foundation clay for the case of weak clay, and in Figure 15 where the failure surfaces move up into the embankment sand for all cases. As there is more certainty regarding the strength of the sand (the coefficient of variations are about 6 percent versus 40 percent for the clay), this amounts to saying that a sand embankment with a low factor of safety can be more reliable than a clay embankment with a higher factor of safety. Similar findings were observed by Wolff (1985) and others.

Review of the relative magnitudes of the variance components indicates that 40 to 48 percent of the problem uncertainty is related to the shear strength of the foundation clay, until the floodwater elevation exceeds 415, at which the contribution of the foundation clay abruptly drops to about 15 percent and then continues to drop as the embankment sand becomes the dominant random variable.

Example Problem 2: Clay Levee on Thick Irregular Clay Top Stratum

Problem modeling

The levee cross section for example problem 2 was illustrated in Figure 4. For slope stability analysis, four random variables were defined; these variables along with their assigned probabilistic moments are shown in Table 11.

Table 11
Random Variables for Example Problem 2

Parameter	Expected Value	Standard Deviation	Coefficient of Variation
Undrained strength of clay levee, c or s_u	800 lb/ft^2	240 lb/ft^2	30%
Undrained strength at top of clay foundation, c or s_u (CPROFL)	500 lb/ft^2	50 lb/ft^2	10%
Rate of increase of undrained strength of clay foundation, (RATEIN)	18 lb/ft^2/ft	2 lb/ft^2/ft	11%
Friction angle of sand foundation, ϕ_{found}	34 deg	2 deg	5.9%

The linearly varying strength option of UTEXAS2 was used to model strength of the clay foundation. The variable CPROFL models the undrained strength at the top of the clay foundation and the variable RATEIN models the rate of increase of the undrained strength with respect to depth. Combination of these two parameters permits the uncertainty in strength to increase with depth. Coefficients of variation were chosen to give a reasonable value for the total uncertainty. Water-filled cracks were specified to a depth of $2c/\gamma$, where the value of c was run-specific.

The piezometric surface in the foundation sands was taken as that obtained for the expected value condition in the underseepage analysis reported in Chapter 6.

Results

The results for all runs for all water elevations are summarized in Table 12. Critical failure surfaces for the cases of floodwater at elevations 400 and 420 are illustrated in Figures 25 and 26. Calculation of the reliability index and probability of failure for each water elevation were accomplished using the spreadsheet templates illustrated in Figures 27 and 28. The resulting conditional probability of failure function is illustrated in Figure 29 and enlarged in Figure 30.

Table 12
Problem 2, Undrained Slope Stability, Results for All Runs

	Material Properties				Water	Run-Time Seed Values				Final Critical Surface			
Run #	c(levee)	c(profl)	rate of c	φ(Found)	Elevation	Xstart	Ystart	Ytanin	Ylimit	FS	XCenter	Ycenter	Radius
101	800	500	18	34	400	30	430	392	300	1.525	31.0	430.8	43.6
102	560	500	18	34	400	30	430	380	300	1.405	28.6	432.6	40.6
103	1040	500	18	34	400	30	430	360	300	1.615	33.4	430.4	47.2
104	800	450	18	34	400	30	430	392	300	1.425	30.8	430.6	43.6
105	800	550	18	34	400	30	430	392	300	1.625	31.2	430.8	43.8
106	800	500	16	34	400	30	430	392	300	1.490	32.0	430.6	45.0
107	800	500	20	34	400	30	430	392	300	1.558	30.4	430.6	42.6
108	800	500	18	32	400	30	430	392	300	1.525	31.0	430.8	43.6
109	800	500	18	36	400	30	430	380	300	1.525	31.0	430.8	43.6
110	800	500	18	34	420	30	430	360	300	1.517	31.2	433.2	47.4
111	560	500	18	34	420	30	430	392	300	1.401	28.6	434.0	42.4
112	1040	500	18	34	420	30	430	392	300	1.603	33.8	433.8	52.6
113	800	450	18	34	420	30	430	392	300	1.419	31.0	433.2	47.0
114	800	550	18	34	420	30	430	392	300	1.616	31.6	433.4	48.0
115	800	500	16	34	420	30	430	392	300	1.481	32.4	433.8	49.4
116	800	500	20	34	420	30	430	392	300	1.552	30.2	433.4	46.0
117	800	500	18	32	420	30	430	392	300	1.517	31.2	433.2	47.4
118	800	500	18	36	420	30	430	392	300	1.517	31.2	433.2	47.4

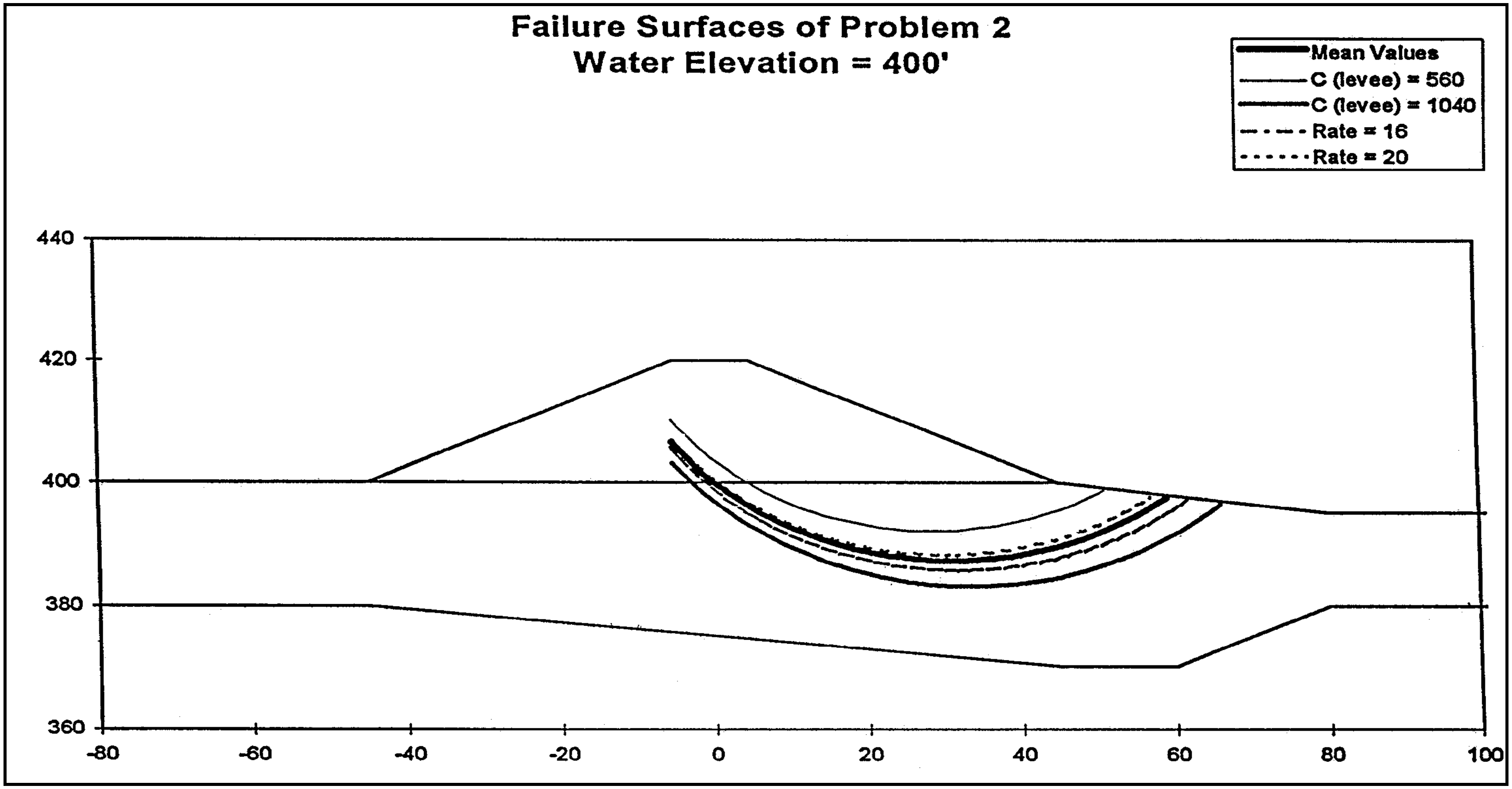

Figure 25. Failure surfaces for example problem 2, water elevation = 400

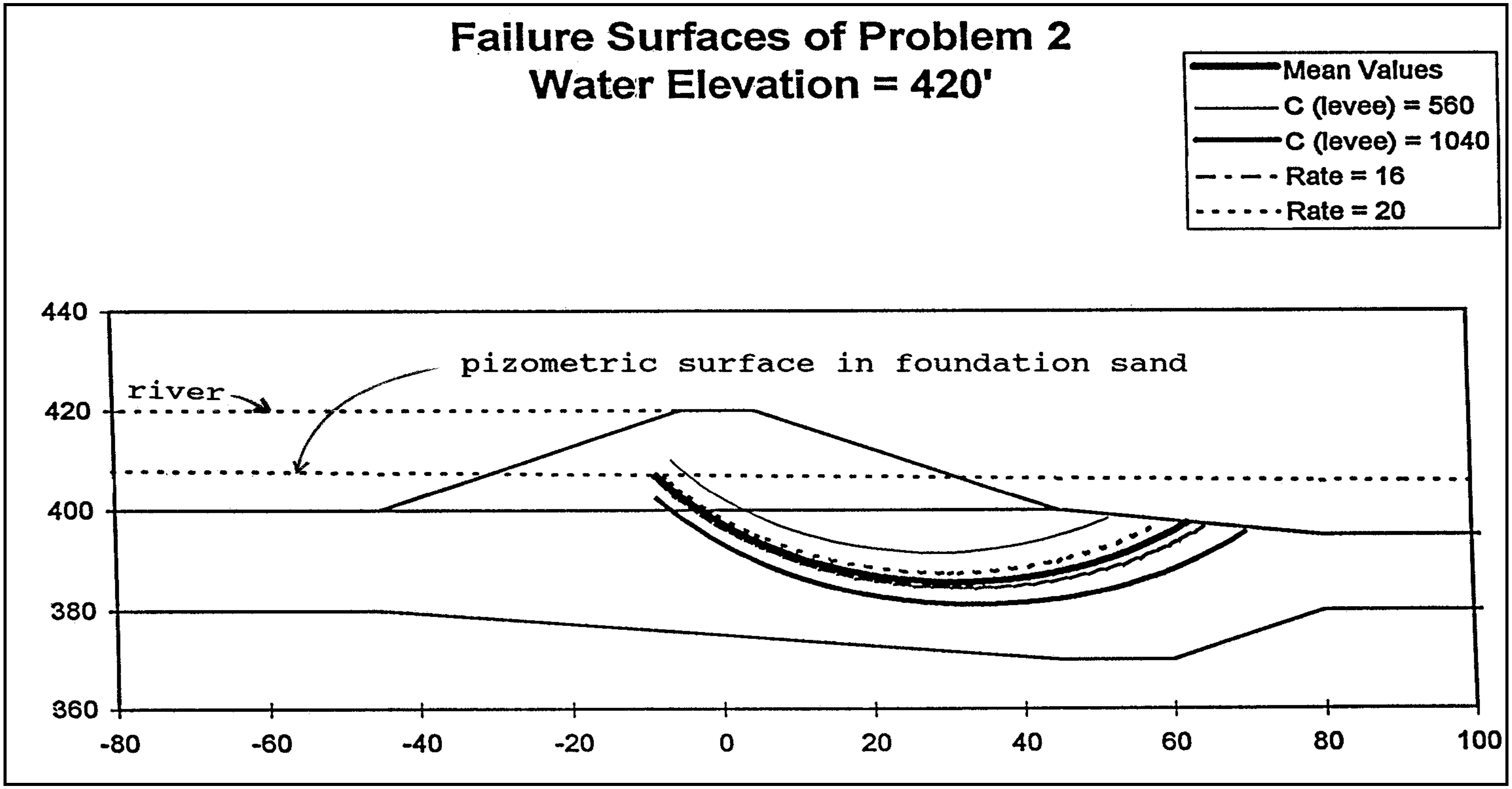

Figure 26. Failure surfaces for example problem 2, water elevation = 420

Slope Stability Analysis
Problem 2

Wolff & Ramon
September 1994

Water Ht = 0
Water El = 400

	c (levee)	c (profl)	rate of c	φ(sub)	FS	Variance componen	% of variance
mean	800	500	18	34	1.525		
	560	500	18	34	1.405		
	1040	500	18	34	1.615	0.011025	49.7047
	800	450	18	34	1.425		
	800	550	18	34	1.625	0.010000	45.0836
	800	500	16	34	1.490		
	800	500	20	34	1.558	0.001156	5.2117
	800	500	18	32	1.525		
	800	500	18	36	1.525	0	0.0000
					Total	0.022181	100.00

E[FS] = 1.525
Var[FS] = 0.022181
sigma[FS] = 0.148933
V(FS) = 9.77%

E[ln FS] = 0.417248
sigma[ln FS] = 0.097429

Beta = 4.2826

FS crit = 1
ln(FS crit) = 0
Pr(f) = 9.244E-06

Figure 27. Reliability calculations for undrained slope stability, example problem 2, water height = 0, water elevation = 400

Slope Stability Analysis
Problem 2

Wolff & Ramon
September 1994

Water Ht = 20
Water El = 420

	c (levee)	c (profl)	rate of c	φ(sub)	FS	Variance component	% of variance
mean	800	500	18	34	1.517		
	560	500	18	34	1.401		
	1040	500	18	34	1.603	0.010201	48.2009
	800	450	18	34	1.419		
	800	550	18	34	1.616	0.009702	45.8443
	800	500	16	34	1.481		
	800	500	20	34	1.552	0.0012603	5.9548
	800	500	18	32	1.517		
	800	500	18	36	1.517	0	0.0000
					Total	0.0211635	100.00

E[FS] = 1.517
Var[FS] = 0.021164
sigma[FS] = 0.145477
V(FS) = 9.59%

E[ln FS] = 0.4121575

sigma[ln FS] = 0.0956783

Beta = 4.307742671

FS crit = 1

ln(FS crit) = 0

Pr(f) = 8.252E-06

Figure 28. Reliability calculations for undrained slope stability, example problem 2, water height = 20, water elevation = 420

Example Problem 2
Conditional probability of slope failure as a function of flood water height H

H	Pr(f)
0	8.25E-06
20	9.24E-06

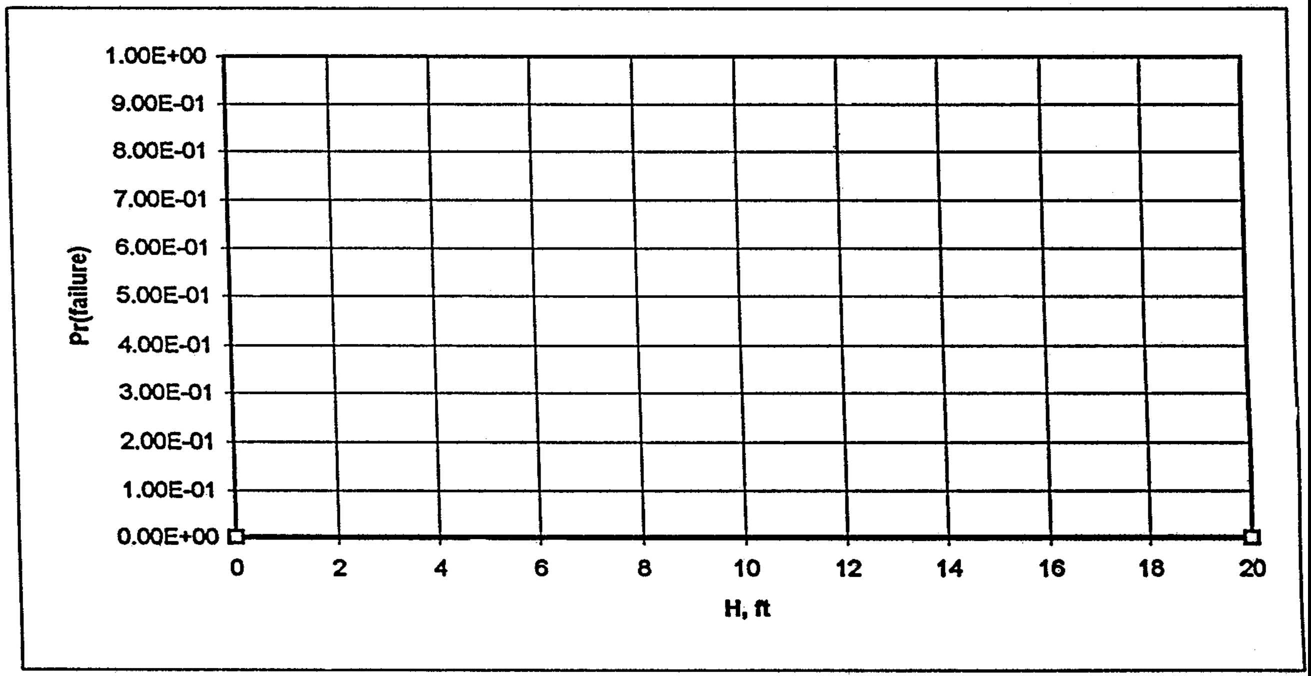

Figure 29. Conditional probability function for undrained slope failure, example problem 2

Example Problem 2
Conditional probability of slope failure as a function of flood water height H

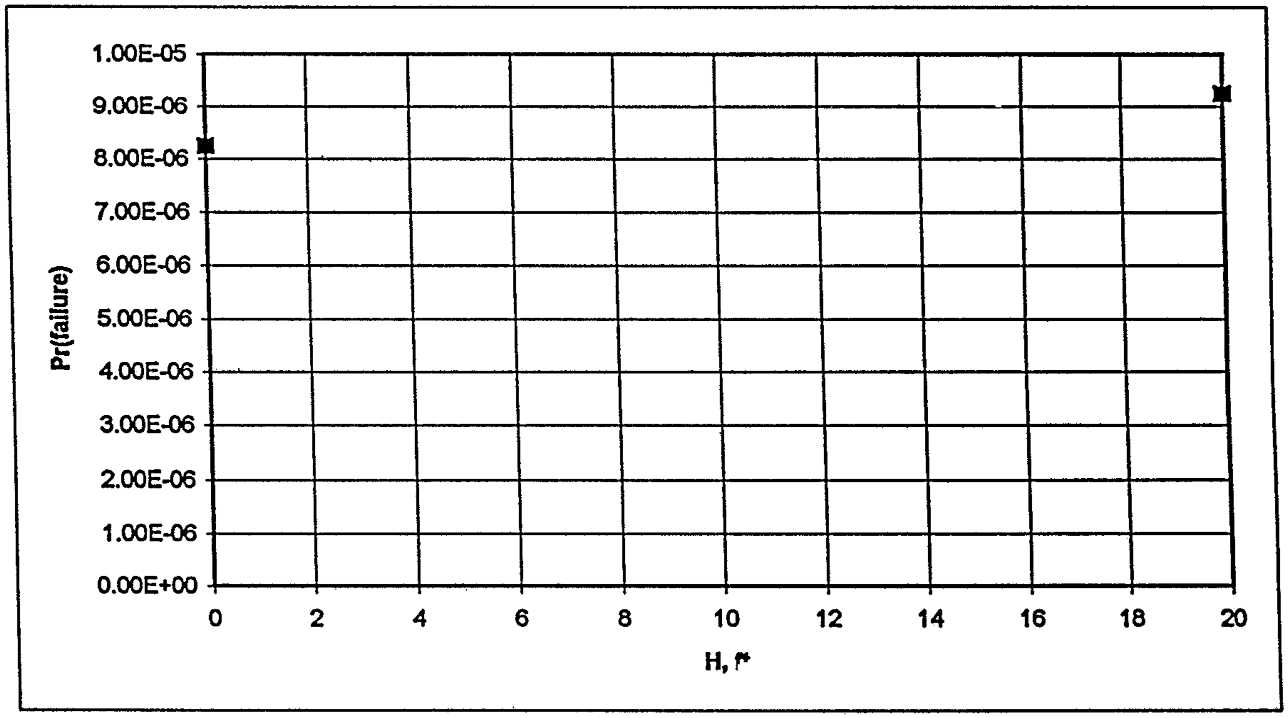

Figure 30. Conditional probability function for undrained slope failure, example problem 2, enlarged view

Discussion

As none of the critical failure surfaces for problem 2 for any of the analysis cases cut into the underlying foundation sands, all of the probability of failure values are low, on the order of 10^{-6}, and are essentially insensitive to floodwater elevation. This is in general agreement with engineering experience; failures of clay slopes are not, in general, related to pool level during the time of inundation. They may, however, be related to pore pressures remaining in an embankment after a flood has receded.

8 Slope Stability Analysis for Long-Term Conditions

"Long-term conditions" are defined as the conditions prevailing at the time when any excess pore pressures due to shear have had sufficient time to dissipate, and stability analyses may be modeled using drained strength parameters in both clay and sand. No examples for slope stability analysis using drained strength parameters for clays are presented in this report. In general, levees subjected to flood loadings would be expected to be loaded for a sufficiently short time that undrained conditions would prevail in clayey materials. Where it is considered that flood durations could be of long enough duration that drained (steady seepage) conditions could develop in clayey embankments or foundations, analyses similar to those in Chapter 7 could be performed. Alternatively, the Taylor's series method could be applied to the infinite slope method of analysis.

As the coefficients of variation for drained strength parameters are typically considerably smaller than those for undrained strength parameters, the probability of failure would be expected to be less than for the undrained case. Wolff (1985) (also cited in Harr (1987)) showed that for well-designed dam embankments, the probability of failure for long-term, steady seepage conditions analyzed using drained strengths can be several orders of magnitude lower than for short-term (after construction) conditions analyzed using undrained strengths.

9 Through-Seepage Analysis

Introduction

Definition

Three types of internal erosion or *piping* can occur as a result of seepage through a levee:

a. If there are cracks in the levee due to hydraulic fracturing, tensile stresses, decay of vegetation, or animal activity along the contours of hydraulic structures, etc., where the water will have a preferential path of seepage, piping may occur. For piping to occur, the tractive shear stress exerted by the flowing water must exceed the critical tractive shear stress of the soil.

b. High exit gradients on the downstream face of the levee may cause piping and possible progressive backward erosion. This is the same phenomenon which was addressed in Chapter 6 and piping occurs when the exit gradient exceeds the critical exit gradient.

c. Internal erosion (suffusion) or removal of fine grains by excessive seepage forces may occur. This type of piping occurs when the seepage gradient exceeds a critical value.

Design practice

Quantitative erosion analyses are not routinely performed for levee design in the Corps of Engineers, although erodibility is implicity considered in the specification of erosion-resistant embankment materials. For design of sand levees, the procedures used by the Rock Island District based on research by Schwartz (1976) do include some elements of erosion analysis. However, the result of the method is to determine the need for providing toe berms according to a semi-empirical criterion rather than to directly determine the threshold of erosion conditions or predict whether erosion will occur. Presumably, some conservatism is present in the berm criteria and thus the criteria do not represent a true *limit state*. Well-constructed clay levees are generally considered resistant to internal erosion, but such erosion can occur where there is a pre-existing crack, defect, or discontinuity and the clay is erodible or dispersive under the effect of a locally high internal gradient. Observed erosion problems in clay embankments have occurred in cases such as poor compaction around drainage culverts and where dispersive clays are present.

Deterministic models

There is no single widely accepted analytical technique or performance function in common use for predicting internal erosion. As probabilistic analysis requires the selection of such a function upon which to calculate probability values, it will be necessary to choose one or two for purposes of illustration herein. Review of various erosion models indicates that erodibility is taken to be a function of some set of the following parameters:

a. Permeability or hydraulic conductivity k.

b. Hydraulic gradient i.

c. Porosity n.

d. Critical stress τ_c (the shear stress required for flowing water to dislodge a soil particle).

e. Particle size, expressed as some representative size such as D_{50} or D_{85}.

f. Friction angle ϕ or angle of repose.

Essentially, the analyses use the gradient, critical tractive stress, and particle size to determine whether the shear stresses induced by seepage head loss are sufficient to dislodge soil particles, and use the gradient, permeability, and porosity to determine whether the seepage flow rate is sufficient to carry away or transport the particles once they have been dislodged. Grain size and pore size information may also be used to determine whether soils, once dislodged, will continue to move (piping) or be caught in the adjacent soil pores (plugging).

It is commonly known that very fine sands and silt-sized materials are among the most erosion-susceptible soils. This arises from their having a critical balance of relatively high permeability, low particle weight, and low critical tractive stress. Particles larger than fine sand sizes are generally too heavy to be moved easily, as particle weight increases with the cube of size. Particles smaller than silts (i.e., clay sizes), although of light weight, may have relatively large electro-chemical forces acting on them, which can substantially increase the critical tractive stress τ_c, and also have sufficiently small permeability as to inhibit particle transport in significant quantity.

The models considered herein to illustrate probabilistic erosion analysis are:

a. Work by Khilar, Folger, and Gray (1985) for clay embankments.

b. The Rock Island District procedure.[1]

c. Extension of the work by Khilar, Folger, and Gray (1985).

In the event that other erosion models are adopted as Corps policy at some later time, or in cases where geotechnical engineers have experience with other erosion models, such models can be substituted for the illustrated methods, using the same approach of defining the probability of failure as the probability that the performance function crosses the limit state.

Erosion model of Khilar, Folger, and Gray

Khilar, Folger, and Gray (1985) investigated the potential for clay soils to pipe or plug under induced flow gradients using a mathematical analysis of a cylindrical opening in the soil. In each element of the cylinder, the tendency for soil dispersion depends on the dissolved solids content of the water (function of the upgradient erosion) and the exchangeable sodium percentage (ESP), where the latter parameter is defined as:

$$ESP = \frac{Na*}{CEC} \times 100\% \qquad (20)$$

In the above equation, *Na** is the exchangeable sodium and *CEC* is the cation exchange capacity.

The tendency for plugging or piping depends on the capability for particle capture at the pore throats. Soil and water samples from Corps of Engineers' Districts throughout the United States were used in laboratory verification studies. Khilar, Folger, and Gray defined two lumped parameters, N_F, and N_G. For erosion to initiate, N_F should initially be greater than N_G, which means that "the initial flow rate should be sufficient to produce a shear stress which is greater than the critical shear stress τ_c for the particular soil-water system." When these parameters are set equal to each other, the following expression for the pressure gradient required to sustain erosion results:

$$\left(\frac{\Delta P}{\Delta L}\right) = \frac{\tau_c}{2.828}\left(\frac{n_0}{K_0}\right)^{1/2} \qquad (21)$$

[1] Personal Communication, 1993, S. Zaidi, U.S. Army Engineer District, Rock Island; Rock Island, IL.

where

$\Delta P/\Delta L$ = pressure gradient in units of pressure per length

τ_c = critical tractive shear stress

n_c = initial porosity

K_o = initial intrinsic permeability in units of length^2 (for water at 20 °C, when k = 1 × 10^{-5} cm/sec, K = 10^{-10} cm^2)

as $\Delta P/\Delta L = i\gamma_w$, the above expression can be rewritten as:

$$i_c = \frac{\tau_c}{2.878\gamma_w}\left(\frac{n_o}{K_o}\right)^{1/2} \tag{22}$$

which provides a measure of the critical gradient required to cause piping.

The critical shear stress τ_c can vary widely, with values for clay ranging from less than 0.2 to more than 20 dynes/cm^2, depending on the soil pore fluid concentration, dielectric dispersion, and sodium absorption ratio. These are parameters not generally available to geotechnical engineers doing preliminary economic analyses of existing levees. However, it can be shown that, in most cases, the gradients required for clay soils are so high as to not be expected in levee embankments and hence the probability of failure due to internal erosion may be small in comparison to other more dominant modes. For example, Khilar, Folger, and Gray (1985) use the following to check the criterion by Arulanandan and Perry (1983) that soil can be considered nonerodible if $\tau_c > 10$ dynes/cm^2.

Assume $n = 0.4$ and $k_o = 10^{-10}$ cm^2 ($k = 10^{-5}$ cm/sec). Then, according to the above equation,

$$i_c = \frac{10 \ \text{dynes/cm}^2}{2.828 \ \ 980.7 \ \text{dynes/cm}^3}\left(\frac{0.4}{10^{-10}\text{cm}^2}\right)^2 = 228 \tag{23}$$

As hydraulic gradients on the order of 200 seldom occur in earth embankments, or in laboratory experiments such as the pinhole test, piping erosion is generally not observed at such for materials with critical tractive stresses as large as 10 dynes/cm^2.

Rock Island District procedure for sand levees

The Rock Island District procedure to ensure the erosion stability of the landside slope of sand levees involves the calculation of two parameters, the maximum erosion susceptibility M and the relative erosion susceptibility R. The calculated values are compared to critical combinations for which toe berms are considered necessary. The parameters are functions of the embankment geometry and soil properties. To analyze stability, first the vertical distance of the seepage exit point on the downstream slope y_c, is determined using the well-known solution for "the basic parabola" by L. Casagrande. Two parameters λ_1 and λ_1 are then calculated as:

$$\lambda_1 = \cos\beta - \frac{\gamma_w}{\gamma_b}\sin\beta\tan(\beta - \delta) - \frac{\gamma_{sat}}{\gamma_b}\frac{\sin\beta}{\tan\phi} \tag{24}$$

$$\lambda_2 = \gamma_w \sin^{0.7}\beta\left(\frac{n}{1.49}\right)^{0.6}[k\tan(\beta - \delta)]^{0.6} \tag{25}$$

where

β = downstream slope angle

δ = zero for a horizontal exit gradient

n = Manning's coefficient for sand, typically 0.02

γ_{sat} = saturated density of the sand in lb/ft^3

γ_b = submerged effective density of the sand in lb/ft^3

k = permeability in ft/s

ϕ = friction angle

It is important to note that the parameter λ_2 is not dimensionless, and the units stated above must be used.

The erosion susceptibility parameters are then calculated as:

$$M = \frac{\lambda_2 y_e^{0.6}}{\lambda_1 \tau_\infty} \tag{26}$$

$$R = \frac{y_e - \left(\frac{\lambda_1 \tau_\infty}{\lambda_2}\right)^{1.67}}{H} \tag{27}$$

In the above equations, τ_{∞} is the critical tractive stress, which the Rock Island District takes as typically about 0.03 lb/ft^2 (14.36 dynes/cm^2) for medium sand, and H is the full embankment height, measured in feet. Again, it should be noted that the parameters M and R values are not dimensionless, and must be calculated using the units shown. According to the Rock Island design criteria, toe berms are recommended when M and R values fall above the shaded region shown in Figure 31. To simplify probabilistic analysis, Shannon and Wilson, Inc., and Wolff (1994) suggested replacing this region with a linear approximation (also shown in Figure 31), and taken to be the limit state. The linear approximation is represented by the following equation:

$$M + 14.4R - 13.0 = 0 \tag{28}$$

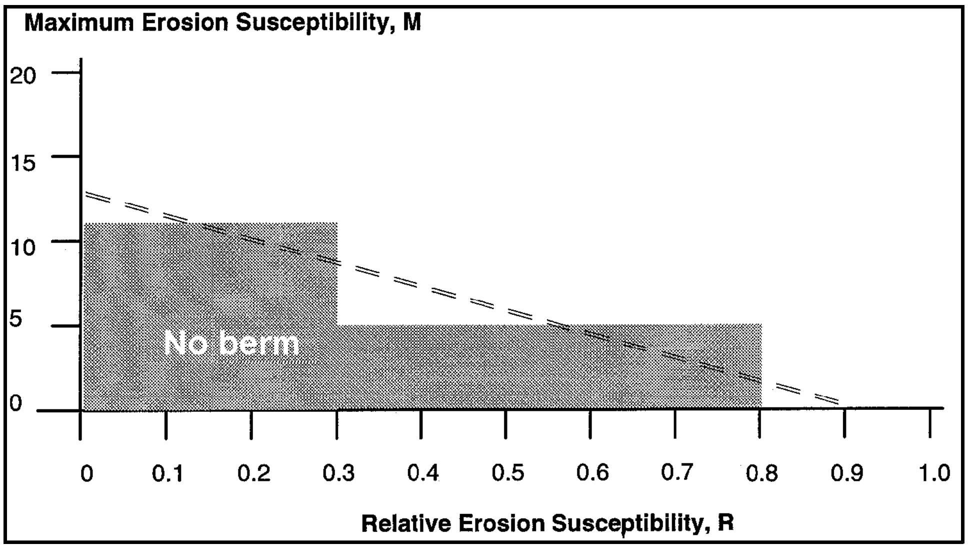

Figure 31. Rock Island District berm criteria and linear approximation of limit state

Positive values of the expression to the left of the equals sign indicate the need for toe berms.

Extension of Khilar's model to sandy materials

Khilar's model was developed for soils with a sufficient cohesive component to sustain an open crack. For these soils, it has been shown that very high gradients, much higher than would typically be found in flood control levees, are necessary to initiate piping.

However, if the same equation given above is considered for silty and sandy materials, reasonable results are obtained that are consistent with engineering expectations of what gradients might initiate piping in such materials. Knowing the D_{50} and D_{10} grain sizes, reasonable estimates of the permeability k and the critical tractive stress τ_c can be made and substituted in Khilar's equation. The critical tractive stress for granular materials can be estimated from the D_{50} size (Lane 1935) as:

$$\tau_c \text{ (dynes/cm}^2\text{)} = 10 \times D_{50} \text{ (in mm)} \quad (29)$$

The permeability k can be estimated from the D_{10} grain size using the well-known correlation developed for Mississippi River levees published in TM3-424 (U.S. Army Corps of Engineers 1956a).

Table 13 summarizes the critical gradients calculated using the above procedure for three granular materials from which a levee might be constructed. It is noted that the relative magnitudes of the calculated critical gradients appear reasonable and this procedure might be considered as a possible approach for initial evaluation of the erodibility of existing granular levees. However, it should also be noted that internal gradients in a pervious levee will generally be below these values, and will seldom exceed 0.20, unless local discontinuities are present.

Table 13
Calculated Critical Gradients for Three Granular Soils Using Khilar's Equation

Soil	D_{50}, mm	τ_c, dynes/cm^2	D_{10}, mm	k, cm/sec	Critical gradient
Uniform fine sand	0.1	1.0	0.09	150×10^{-4}	0.59
Silty gravelly sand	0.4	4.0	0.005	10×10^{-4}	9.1
Coarse to medium sand	1.8	18.0	0.3	$2{,}000 \times 10^{-4}$	2.9

Example Problem 1: Sand Levee on Thin Uniform Clay Top Stratum

The erosion resistance of example problem 1 will be evaluated using two techniques, as follows:

a. The Rock Island criteria.

b. The extended Khilar model.

The embankment soil will be taken to be a coarse-to-medium sand similar to that in the third row of Table 13. Random variables are characterized as shown in Table 14.

The analysis for the Rock Island method and the Khilar equation method was performed using a spreadsheet extended from one previously developed by Shannon and Wilson, Inc., and Wolff (1994). An example of the spreadsheet is shown in Figure 32.

Table 14
Random Variables for Internal Erosion Analysis, Example Problem 1

Variable	Expected Value	Coefficient of Variation	Rock Island Model	Khilar's Model
Mannings coefficient, n	0.02	10%	*	
Unit weight, γ_{sat}	125 lb/ft^3	8%	*	
Friction angle, ϕ	30 deg	6.7%	*	
Coefficient of permeability, *k*	2,000 × 10^{-4} cm/s	30%	*	*
Critical tractive stress, τ	18 dynes/cm^2	10%	*	*

Rock Island District method

For the Rock Island District method, which assesses erosion at the landside seepage face, the method was numerically unstable (λ_1 becomes negative) for the slopes assumed in example problem 1. To make the problem stable, the slopes had to be flattened to 1V:3H riverside and 1V:5H landside.

The results for the Taylor series analysis for a 20-ft water height are summarized in Table 15. Results for other heights are shown in the spreadsheets in Figures 33 through 37.

Table 15
Results of Internal Erosion Analysis, Example Problem 1 (Modified to Flatter Slopes) H = 20 ft, Rock Island District Method

n	γ_{set} lb/ft^3	ϕ	k × 10^{-4} cm/sec	τ_c dynes/cm^2	Performance Function	Variance Component	Percent of Total Variance
0.02	125	30	2000	18	17.524		
0.022	125	30	2000	18	18.491		
0.018	125	30	2000	18	16.515	0.9761	2.1
0.02	135	30	2000	18	14.798		
0.02	115	30	2000	18	23.667	19.6648	42.9
0.02	125	32	2000	18	14.817		
0.02	125	28	2000	18	22.179	13.5498	29.5
0.02	125	30	2600	18	20.321		
0.02	125	30	1400	18	14.339	8.961	19.5
0.02	125	30	2000	19.8	16.046		
0.02	125	30	2000	16.2	19.369	2.7606	6.0

Through Seepage Analysis by Schwartz / Rock Island Method
Piping Analysis by Khilar Equation
Example Problem 1
Expected Value Case

Slope Geometry Parameters

Downstream Slope, I on	5.00
Slope angle, beta, degrees	11.31
Slope angle, beta, radians	0.20
Upstream Slope, 1 on	3.00
Slope angle, beta, degrees	18.43
Slope angle, beta, radians	0.32
Crown width, ft	10.00
Embankment height, H	20.00
Water height, h	10.00
Base Width, ft	170.00
Length of slope under water, m, ft	50.00
basic parabola, d, ft	135.00
S sub o, ft	135.37
a, ft	9.97
Exit point y sub e, ft	1.96

Exit gradient parameters

Exit gradient orientation, delta, degrees	0
Exit gradient orientation, delta, radians	0

Sand parameters

Manning's roughness for sand	0.02
Gamma sat for sand, pcf	125
Gamma buoyant, pcf	62.6
Friction angle, phi, degrees	30
Friction angle, phi, radians	0.523599
Sand permeability, k, cm/sec	0.2
Sand permeability, k, ft/sec	0.006562
Intrinsic permeability, K, cm^2	0.000002
porosity	0.4
tau sub c, psf	0.0396
tau sub c, dynes /cm^2	18.00

Rock Island Results

lambda(1) =	0.26
lambda(2) =	0.0280
lambda(1)/lambda(2) =	9.39
M =	4.02
R =	0.09
M + 14.44R -13.0 =	-7.707

Internal Erosion Model

Embankment

seep drop =	8.04
length =	110.22
i =	0.073
i crit =	2.903
FS =	39.78

Figure 32. Spreadsheet for through-seepage analysis

It is noted that the most significant random variables, based on descending order of their variance components, are the unit weight, the friction angle, and the permeability. The effects of Manning's coefficient and the critical tractive stress, at least for the coefficients of variation assumed, are relatively insignificant.

Internal Erosion Analysis
Rock Island Method, 1 on 3 RS, 1on 5 LS
file:tsts01.xls

T. F. Wolff
September 1994

h = 5

	n	gamma sat	phi	k	tau	perf function	Variance component	% of variance
mean	0.02	125	30	2000.00	18.00	-11.236		
	0.022	125	30	2000.00	18.00	-11.129		
	0.018	125	30	2000.00	18.00	-11.349	0.0121	2.16
	0.02	135	30	2000.00	18.00	-11.549		
	0.02	115	30	2000.00	18.00	-10.579	0.2352	42.06
	0.02	125	32	2000.00	18.00	-11.546		
	0.02	125	28	2000.00	18.00	-10.734	0.1648	29.47
	0.02	125	30	2600.00	18.00	-10.931		
	0.02	125	30	1400.00	18.00	-11.603	0.1129	20.19
	0.02	125	30	2000.00	19.80	-11.403		
	0.02	125	30	2000.00	16.20	-11.033	0.0342	6.12
						Total	0.5593	100.00

E[PF] =	-11.236		
Var[PF] =	0.559282	beta =	15.024
sigma[PF] =	0.747852	Pr(fail) =	0

Figure 33. Reliability calculations for through-seepage, example problem 1, *h* = 5 ft

Internal Erosion Analysis
Rock Island Method, 1 on 3 RS, 1on 5 LS
file:tsts02.xls

T. F. Wolff
September 1994

h = 10

	n	gamma sat	phi	k	tau	perf function	Variance component	% of variance
mean	0.02	125	30	2000.00	18.00	-7.703		
	0.022	125	30	2000.00	18.00	-7.457		
	0.018	125	30	2000.00	18.00	-7.968	0.0653	2.14
	0.02	135	30	2000.00	18.00	-8.419		
	0.02	115	30	2000.00	18.00	-6.141	1.2973	42.56
	0.02	125	32	2000.00	18.00	-8.414		
	0.02	125	28	2000.00	18.00	-6.518	0.8987	29.49
	0.02	125	30	2600.00	18.00	-6.989		
	0.02	125	30	1400.00	18.00	-8.541	0.6022	19.76
	0.02	125	30	2000.00	19.80	-8.091		
	0.02	125	30	2000.00	16.20	-7.232	0.1845	6.05
						Total	3.0480	100.00

E[PF] = -7.703
Var[PF] = 3.047952
sigma[PF] = 1.745838

beta = 4.412
Pr(fail) = 5.12E-06

Figure 34. Reliability calculations for through-seepage, example problem 1, *h* = 10 ft

Internal Erosion Analysis
Rock Island Method, 1 on 3 RS, 1on 5 LS
file:tsts03.xls

T. F. Wolff
September 1994

h = **15**

	n	gamma sat	phi	k	tau	perf function	Variance component	% of variance
mean	**0.02**	**125**	30	2000.00	18.00	**-1.700**		
	0.022	125	30	2000.00	18.00	-1.250		
	0.018	125	30	2000.00	18.00	-2.180	0.2162	2.16
	0.02	**135**	30	2000.00	18.00	-2.980		
	0.02	**115**	30	2000.00	18.00	1.150	4.2642	42.56
	0.02	125	**32**	2000.00	18.00	-2.980		
	0.02	125	**28**	2000.00	18.00	0.466	2.9687	29.63
	0.02	125	30	**2600.00**	18.00	-0.398		
	0.02	125	30	**1400.00**	18.00	-3.200	1.9628	19.59
	0.02	125	30	2000.00	19.80	-2.400		
	0.02	125	30	2000.00	16.20	-0.841	0.6076	6.06
						Total	10.0196	100.00

E[PF] =	-1.700		
Var[PF] =	10.0196	beta =	0.537
sigma[PF] =	3.165375	Pr(fail) =	0.295613

Figure 35. Reliability calculations for through-seepage, example problem 1, *h* = 15 ft

Internal Erosion Analysis
Rock Island Method, 1 on 3 RS, 1on 5 LS
file:tsts04.xls

T. F. Wolff
September 1994

h = 17.5

	n	gamma sat	phi	k	tau	perf function	Variance component	% of variance
mean	0.02	125	30	2000.00	18.00	3.319		
	0.022	125	30	2000.00	18.00	3.921		
	0.018	125	30	2000.00	18.00	2.691	0.3782	2.13
	0.02	135	30	2000.00	18.00	1.619		
	0.02	115	30	2000.00	18.00	7.130	7.5928	42.79
	0.02	125	32	2000.00	18.00	1.632		
	0.02	125	28	2000.00	18.00	6.208	5.2349	29.50
	0.02	125	30	2600.00	18.00	5.057		
	0.02	125	30	1400.00	18.00	1.332	3.4689	19.55
	0.02	125	30	2000.00	19.80	2.399		
	0.02	125	30	2000.00	16.20	4.466	1.0681	6.02
						Total	17.7430	100.00

E[PF] = 3.319
Var[PF] = 17.74298
sigma[PF] = 4.212241

beta = -0.788
Pr(fail) = 0.784635

Figure 36. Reliability calculations for through-seepage, example problem 1, *h* = 17.5 ft

Internal Erosion Analysis
Rock Island Method, 1 on 3 RS, 1on 5 LS
file:tsts05.xls

T. F. Wolff
September 1994

h = 20

	n	gamma sat	phi	k	tau	perf function	Variance component	% of variance
mean	0.02	125	30	2000.00	18.00	17.524		
	0.022	125	30	2000.00	18.00	18.491		
	0.018	125	30	2000.00	18.00	16.515	0.9761	2.13
	0.02	135	30	2000.00	18.00	14.798		
	0.02	115	30	2000.00	18.00	23.667	19.6648	42.85
	0.02	125	32	2000.00	18.00	14.817		
	0.02	125	28	2000.00	18.00	22.179	13.5498	29.52
	0.02	125	30	2600.00	18.00	20.321		
	0.02	125	30	1400.00	18.00	14.339	8.9461	19.49
	0.02	125	30	2000.00	19.80	16.046		
	0.02	125	30	2000.00	16.20	19.369	2.7606	6.01
						Total	45.8974	100.00

E[PF] =	17.524		
Var[PF] =	45.89736	beta =	-2.587
sigma[PF] =	6.774759	Pr(fail) =	0.995154

Figure 37. Reliability calculations for through-seepage, example problem 1, *h* = 20.0 ft

When the probabilities of failure from the individual spreadsheet solutions are plotted, the result is the conditional probability of failure function shown in Figure 38. Again, it takes the expected reverse-curve shape. Below heads of 10 ft, or about half the levee height, the probability of failure against through-seepage failure is virtually nil. The probability of failure becomes greater than 0.5 for a head of about 16.5 ft, and approaches unity at the full head of 20 ft.

Khilar equation

The analysis was repeated using the original geometry for example problem 1 and using Equation 21 to predict the critical gradient for piping. The actual gradient was estimated as the head loss from the riverside water elevation to the landside slope exit point (based on the basic parabola) divided by the horizontal distance between these two points. The factor of safety was taken as the critical gradient divided by the actual gradient. As shown in the spreadsheets in Figure 39, the reliability index values were greater than 12, even for a full head on the levee, corresponding to a nil ($<10^{-6}$) probability of failure.

Example Problem 2: Clay Levee on Thick Non-uniform Clay Top Stratum

For any reasonable values of the critical tractive stress and permeability for clays, the calculated factors of safety were extremely large, indicating that the probability of failure against piping would be nil in well-constructed clay embankments. It is understood that piping may still occur at undetected areas of poor construction or defects, but analytical models for such conditions are not available, requiring that probability values be estimated judgmentally or based on historical data.

Example Problem 1 – 1 on 3 RS, 1 on 5 LS, Rock Island Criteria

Conditional Probability of Through Seepage Failure as a function of flood water height H

H	Pr(f)
0.0	0
5.0	0
10.0	0.000
15.0	0.296
17.5	0.785
20.0	0.995

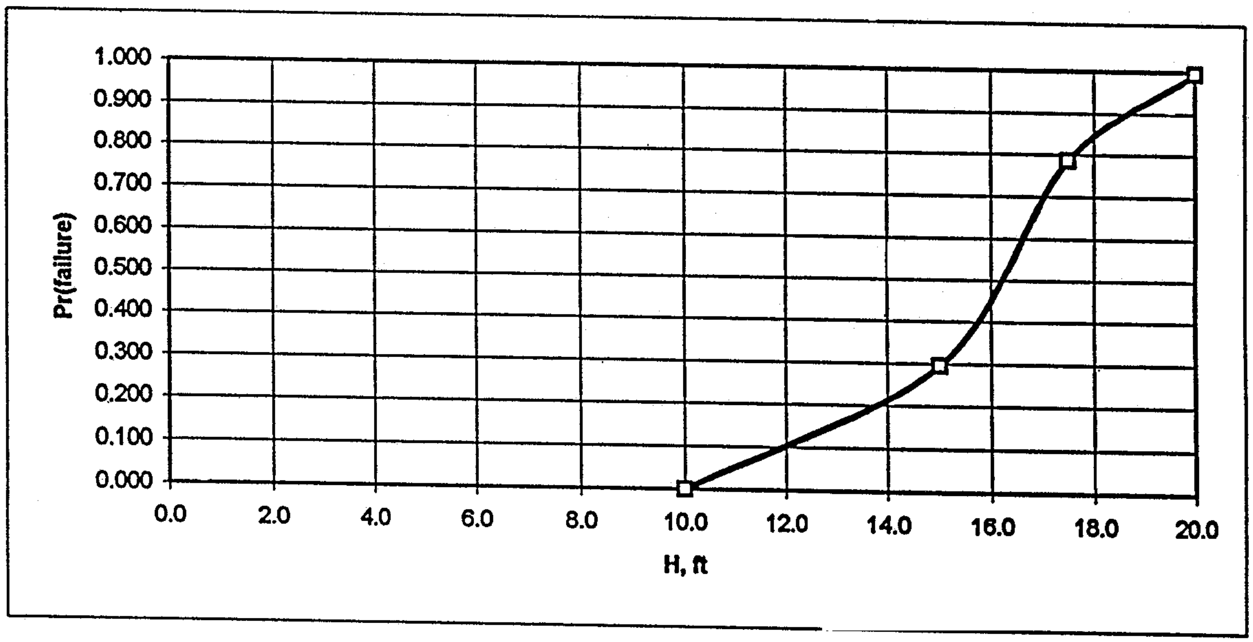

Figure 38. Conditional probability of failure function for through-seepage, example problem 1 (modified)

Internal Erosion Analysis
Khilar Equation, Example Problem 1
file:tsts10.xls

T. F. Wolff
September 1994

H = 20

	k	tau	FS	Variance component	% of variance
mean	2000.00	18.00	9.690		
	2600.00	18.00	8.500		
	1400.00	18.00	11.580	2.37	71.60
	2000.00	19.80	10.650		
	2000.00	16.20	8.710	0.94	28.40
			Total	3.31	100.00

E[FS] =	9.690	Var[lnFS] =	0.034670333		
Var[FS] =	3.3125	sigma[lnFS] =	0.186199713	beta =	12.104
sigma[FS] =	1.820027472	E[lnFS] =	2.253759259	Pr(fail) =	0
V(FS) =	0.187825333				

Figure 39. Reliability calculations for internal erosion analysis using modified Khilar's equation

10 Surface Erosion

Introduction

As flood stages increase, the potential increases for surface erosion from the following two sources:

a. Erosion due to excessive current velocities parallel to the levee slope.

b. Erosion due to wave attack directly against the levee slope.

The Corps of Engineers provides protection against these events for new construction by providing adequate slope protection, typically a thick grass cover for most levees, and stone revetment at locations expected to be susceptible to wave attack. During flood emergencies, additional protection may be provided where necessary using dumped rock, snow fence, or plastic sheeting.

Erosion Due to Current Velocity

Analytical model

Although there are criteria for decision-making relative to the need for slope protection and the design of slope protection, they are not in the form of a limit state or performance function (i.e., one does not typically calculate a factor of safety against scour). To perform a reliability analysis, one needs to define the problem as a comparison between the probable velocity and the velocity that will result in damaging scour. Considerable research could be undertaken to derive an appropriate model. As a first approximation for the purpose of illustration, this chapter will use a simple adaptation of Manning's formula for average flow velocity and assume that the critical velocity for a grassed slope can be expressed by its expected value and coefficient of variation.

Velocity. For channels that are very wide relative to their depth (width $\geq$ 10×depth), the velocity can be expressed as:

$$V = \frac{1.486 y^{2/3} S^{1/2}}{n} \tag{30}$$

where

y = depth of flow

S = slope of the energy line

n = Manning's roughness coefficient

For the purpose of illustration, it will be assumed that the velocity of flow parallel to a levee slope for water heights from 0 to 20 ft can be approximated using the above formula with y taken from 0 to 20 ft. For real levees in the field, it is likely that better estimates of flow velocities at the location of the riverside slope can be obtained by more detailed hydraulic models (see EM1110-2-1418 (U.S. Army Corps of Engineers 1994)).

For purposes of illustration, the following probabilistic moments are assumed. More detailed and site-specific studies would be necessary to determine appropriate values.

$$E[S] = 0.0001 \qquad V_s = 10\% \tag{31}$$

$$E[n] = 0.03 \qquad V_n = 10\% \tag{32}$$

Critical velocity. For purposes of illustration, it is assumed that the critical velocity that will result in damaging scour can be expressed as:

$$E[V_{crit}] = 5.0\ ft/\sec \quad V_{vcrit} = 20\% \tag{33}$$

Further research is necessary to develop guidance on appropriate values for prototype structures.

Calculation of reliability index and probability of failure

The Manning equation is of the form

$$G(x_1, x_2, x_3, \ldots) = a x_1^{g1} x_2^{g2} x_3^{g3} \tag{34}$$

For equations of this form, Harr (1987) shows that the probabilistic moments can be easily determined using a special form of the Taylor's series approximation he refers to as the *vector equation*. In such cases, the expected value of the function is evaluated as the function of the expected values. The coefficient of variation of the function can be calculated as:

$$V_G^2 = g_1^2 V^2(x_1) + g_2^2 V^2(x_2) + g_3^2 V^2 x_3) + \ldots \tag{35}$$

For the case considered, the coefficient of variation of the flow velocity is then:

$$V_v = \sqrt{V_n^2 + \left(\frac{1}{4}\right) V_s^2} \tag{36}$$

Note that, although the velocity increases with floodwater height *y*, the coefficient of variation of the velocity is constant for all heights.

Knowing the expected value and standard deviation of the velocity and the critical velocity, a performance function can be defined as the ratio of critical velocity to the actual velocity, (i.e., the factor of safety) and the limit state can be taken as this ratio equaling the value 1.0. If the ratio is assumed to be lognormally distributed as described in Annex A, then the reliability index is:

$$\beta = \frac{1n\left(\frac{E[C]}{E[D]}\right)}{\sqrt{V_C^2 + V_D^2}} = \frac{1n\left(\frac{E[V_{crit}]}{E[V]}\right)}{\sqrt{V_{vcrit}^2 + V_v^2}} \tag{37}$$

and the probability of failure can be determined from the cumulative distribution function for the normal distribution.

Results

The assumed model and probabilistic moments were used to construct the example spreadsheet in Figure 40, which calculates expected values and standard deviations of the flow velocity, the reliability index, and the probability of failure, all as functions of the flood water height y. It is again observed that a typical levee may be highly reliable for water levels up to about one-half the height, and then the probability of failure may increase rapidly.

Demonstration Spreadsheet for Surface Erosion Analysis

		Expected Value		Coefficient of variation	Standard Deviation
Slope	S =	0.0001	V(s) =	0.1	0.00001
Manning's roughness	n =	0.03	V(n) =	0.1	0.003
Scouring velocity	V crit =	5	V(Vcrit) =	0.2	1

Water height, y	E[V] ft/sec	V(v)	sigma v	ln(E[v]/E[Vcrit])	sqr(V^2+V^2)	beta (log)	Pr(failure)
0	0.000	0.1118	0.000				
1	0.495	0.1118	0.055	2.312	0.22912878	10.09	0
2	0.786	0.1118	0.088	1.850	0.22912878	8.07	3.33E-16
3	1.030	0.1118	0.115	1.580	0.22912878	6.89	2.73E-12
4	1.248	0.1118	0.140	1.388	0.22912878	6.06	6.97E-10
5	1.448	0.1118	0.162	1.239	0.22912878	5.41	3.2E-08
6	1.636	0.1118	0.183	1.117	0.22912878	4.88	5.39E-07
7	1.813	0.1118	0.203	1.015	0.22912878	4.43	4.75E-06
8	1.981	0.1118	0.222	0.926	0.22912878	4.04	2.67E-05
9	2.143	0.1118	0.240	0.847	0.22912878	3.70	0.000109
10	2.299	0.1118	0.257	0.777	0.22912878	3.39	0.000349
11	2.450	0.1118	0.274	0.713	0.22912878	3.11	0.000925
12	2.596	0.1118	0.290	0.655	0.22912878	2.86	0.002116
13	2.739	0.1118	0.306	0.602	0.22912878	2.63	0.004302
14	2.877	0.1118	0.322	0.553	0.22912878	2.41	0.007937
15	3.013	0.1118	0.337	0.507	0.22912878	2.21	0.013516
16	3.145	0.1118	0.352	0.464	0.22912878	2.02	0.021523
17	3.275	0.1118	0.366	0.423	0.22912878	1.85	0.032383
18	3.402	0.1118	0.380	0.385	0.22912878	1.68	0.046423
19	3.527	0.1118	0.394	0.349	0.22912878	1.52	0.063847
20	3.650	0.1118	0.408	0.315	0.22912878	1.37	0.084718

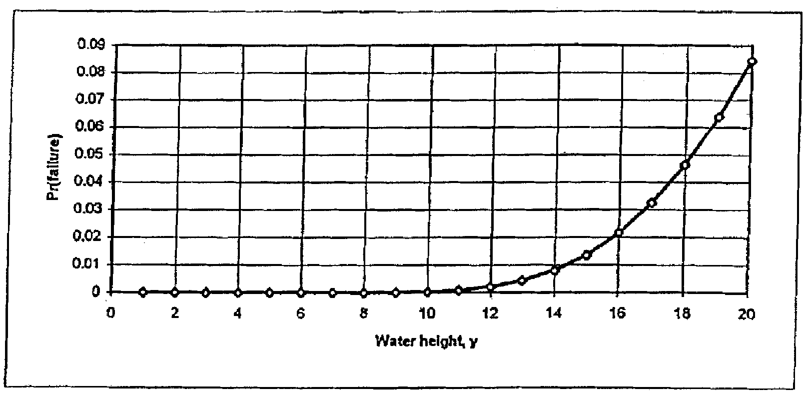

Figure 40. Example spreadsheet for surface erosion analysis

Erosion Due to Wind-Generated Waves

The height and frequency of wind-generated waves are dependent on wind speed, duration of the wind, fetch (over-water distance wind travels while generating waves), and depth of water. As flood stages increase, the potential for wave attack increases due to the increase in fetch and depth of water. The relative effect of wave-caused erosion is highly site-specific, and will vary significantly depending on such factors as direction of exposure to wind waves, whether timber stands exist to shield the levee from wave attack, steepness of the levee slope, and nature of the embankment material.

Wave-caused erosion during prolonged flooding has occurred on the upper Mississippi River where appreciable fetch exists. This is especially a problem in the Rock Island District where levees are constructed of dredged sand and to a lesser degree in the St. Louis District at locations where specific site conditions favorable to wave-caused erosion are present.

Wave-caused erosion is a complicated problem and has not at this time been reduced to an appropriate model which could be used to perform a reliability analysis.

11 Combining Conditional Probability Functions and Other Considerations

Combining Probability Functions

Once a conditional probability of failure function has been obtained for each considered failure mode, it is desired to combine them to determine the total conditional probability of failure of all modes combined as a function of the floodwater elevation (FWE).

As a first approximation, it may be assumed that each of the following four failure modes are independent and hence uncorrelated:

a. Underseepage.

b. Slope stability.

c. Through-seepage and internal erosion.

d. Surface erosion.

This assumption is not necessarily true, as some of the conditions increasing the probability of failure for one mode may likely increase the probability of failure by another. However, there is insufficient research to better quantify such possible correlation, and it is beyond the scope of the present project. Assuming independence considerably simplifies the mathematics involved, which is also a desired condition for studies at the level of economic analysis.

For **underseepage**, the probability of failure at each water elevation is taken as that determined in Chapter 6; i.e., the probability of developing an upward gradient sufficient to cause boiling throughout the top stratum.

For **slope stability**, the probability of failure is taken as the probability that the factor of safety is less than unity, and it is assumed that the factor of safety is lognormally distributed. It is necessary to determine whether modeling short-term conditions only is sufficient, or whether it is necessary to also model long-term conditions and post-flood conditions in the analysis. For the two examples given, only short-term analyses are considered; however, the probability of failure could also be evaluated for these other cases using the same techniques. In such cases, they would not be combined with other failure modes as illustrated in this section, as they are not concurrent events.

For **through-seepage and internal erosion**, the results of the Rock Island District method will be used herein for example 1. The probability of failure is taken as the probability that a function for which a zero value approximates the Rock Island berm criteria in fact assumes a negative value. The performance function is assumed to be normally distributed. It should be recalled that the assumed slopes had to be flattened to make the method numerically stable and the resulting conditional probability of failure function is thus not for the same levee section as those for other modes. It is retained for illustrative purposes to show how probability functions can be combined. For the assessment of internal erosion based on the Khilar, Folger, and Gray (1985) piping model, the probabilities of failure appear to be so low as to be negligible.

For **surface erosion**, a conceptual example based on the Manning equation for flow velocity was illustrated for this report. Additional research needs to be performed to determine the most appropriate way to model the probability of surface erosion, for both current and wave attack, considering the current state-of-the-practice in the Corps of Engineers.

Judgmental evaluation

It is required that a levee under consideration be field inspected. During such an inspection, it is likely that the inspection team may encounter any number of items and features, in addition to the three to four quantified failure modes, that may compromise the confidence of the levee section during a flood event. These might include animal burrows, cracks, roots, and poor maintenance that might impede detection of defects or execution of flood-fighting activities. To provide a mathematical means to factor in such information, one may develop a judgment-based conditional probability function by answering the following question:

> *Discounting the likelihood of failure accounted for in the quantitative analyses, but considering observed conditions, what would an experienced levee engineer consider the probability of failure of this levee for a range of water elevations?*

For the two example problems considered herein, the functions listed in Table 16 were assumed. While this may appear to be “outright guessing,” leaving out such information has the greater danger of not considering the obvious. Formalized techniques for quantifying expert opinion (such as the Delphi method) exist and merit further research for application to the economic analysis of existing levees and existing structures.

+Table 16
Assigned Conditional Probability of Failure Functions for Judgmental Evaluation of Observed Conditions

Floodwater Elevation	Probability of Failure Example 1	Probability of Failure Example 2
400.0	0	0
405.0	0.01	0.005
410.0	0.02	0.01
415.0	0.20	0.02
417.5	0.40	0.05
420.0	0.80	0.10

Combinatorial probabilities

For N independent failure modes, the reliability, or probability of no failure involving any mode, is the probability of no failure due to mode 1 *and* no failure due to mode 2, *and* no failure due to mode 3, etc. As *and* implies multiplication, the overall reliability at a given floodwater elevation is the product of the modal reliability values for that flood elevation, or:

$$R = R_{US} R_{SS} R_{TS} R_{SE} R_J \quad (38)$$

where the subscripts refer to the identified failure modes. Hence the probability of failure at any floodwater elevation is:

$$\begin{aligned} Pr(f) &= 1 - R \\ &= 1 - (1-p_{US})(1-p_{SS})(1-p_{TS})(1-p_{SE})(1-p_J) \end{aligned} \quad (39)$$

The total conditional probability of failure functions calculated for the two example problems are shown in Figures 41 and 42. It is observed that probabilities of failure are generally quite low for water elevations less than one-half the levee height, then rise sharply as water levels approach the levee crest. While there are insufficient data to judge whether this is a general trend for all levees, it has some basis in experience and intuition.

Conditional Probability of Failure Function for Example Problem 1

Flood Water Elevation	Underseepage P(f)	Underseepage R	Slope Stability P(f)	Slope Stability R	Through Seepage P(f)	Through Seepage R	Surface Erosion P(f)	Surface Erosion R	Judgment P(f)	Judgment R	Combined P(f)	Combined R
400.0	0.00000	1.00000	0.00001	0.99999	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000	0.00001	0.99999
405.0	0.00008	0.99992	0.00001	0.99999	0.00000	1.00000	0.00000	1.00000	0.01000	0.99000	0.01009	0.98991
410.0	0.00015	0.99985	0.00002	0.99998	0.00000	1.00000	0.00035	0.99965	0.02000	0.98000	0.02051	0.97949
415.0	0.50000	0.50000	0.00002	0.99998	0.29600	0.70400	0.01352	0.98648	0.20000	0.80000	0.72221	0.27779
417.5	0.70000	0.30000	0.00000	1.00000	0.78500	0.21500	0.03900	0.96100	0.40000	0.60000	0.96281	0.03719
420.0	0.87100	0.12900	0.30870	0.69130	0.99500	0.00500	0.08470	0.91530	0.80000	0.20000	0.99992	0.00008

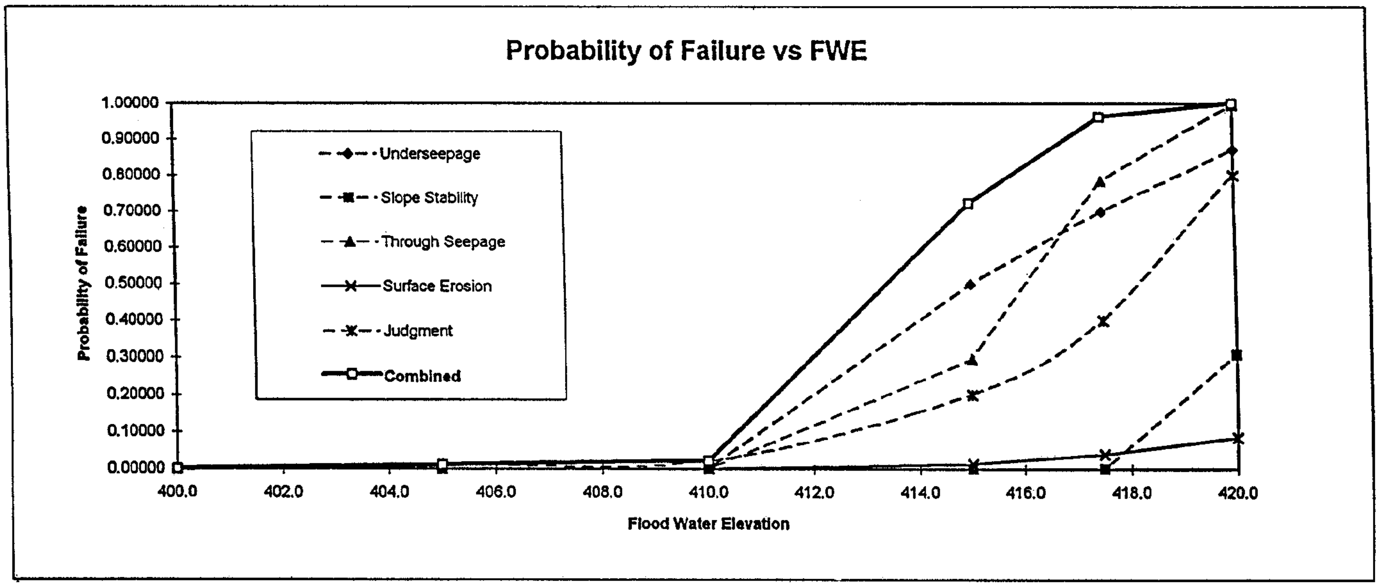

Figure 41. Combined conditional probability of failure function for example problem 1

Conditional Probability of Failure Function for Example Problem 2

Flood Water Elevation	Underseepage P(f)	Underseepage R	Slope Stability P(f)	Slope Stability R	Through Seepage P(f)	Through Seepage R	Surface Erosion P(f)	Surface Erosion R	Judgment P(f)	Judgment R	Combined P(f)
400.0	0.00000	1.00000	0.00001	0.99999	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000	0.00001
405.0	0.00000	1.00000	0.00001	0.99999	0.00000	1.00000	0.00000	1.00000	0.00500	0.99500	0.00501
410.0	0.00000	1.00000	0.00001	0.99999	0.00000	1.00000	0.00035	0.99965	0.01000	0.99000	0.01036
415.0	0.00010	0.99990	0.00001	0.99999	0.00000	1.00000	0.01352	0.98648	0.02000	0.98000	0.03335
417.5	0.00642	0.99358	0.00001	0.99999	0.00000	1.00000	0.03900	0.96100	0.05000	0.95000	0.09292
420.0	0.07830	0.92170	0.00001	0.99999	0.00000	1.00000	0.08470	0.91530	0.10000	0.90000	0.24074

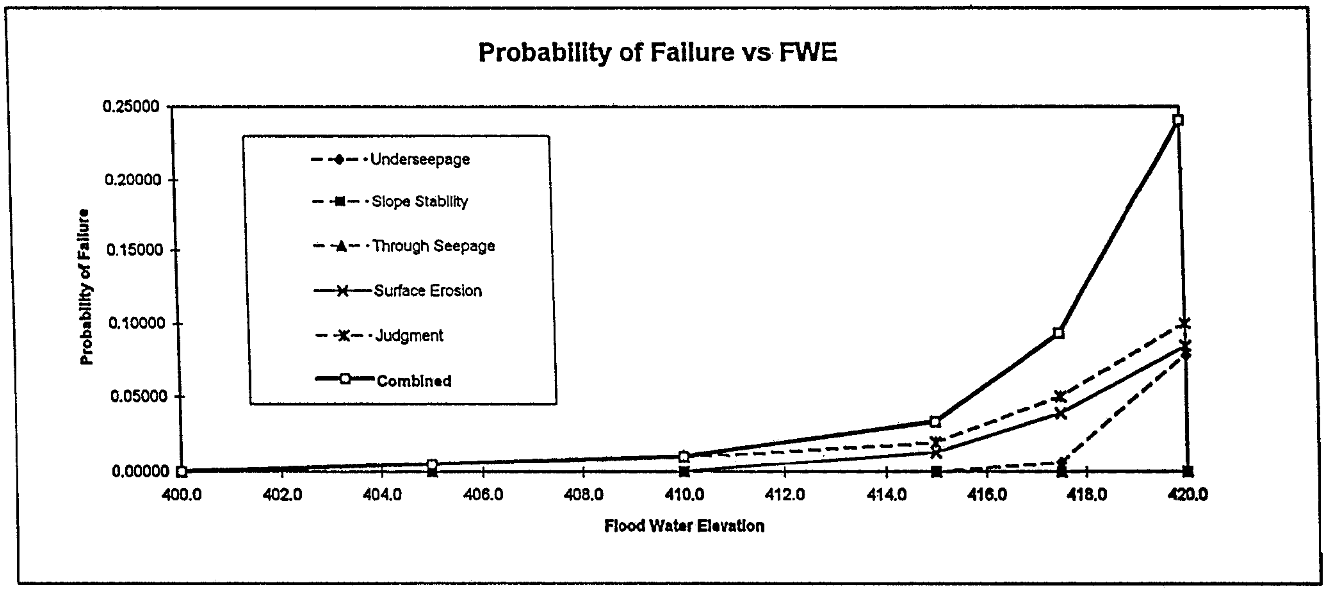

Figure 42. Combined conditional probability of failure function for example problem 2

Flood Duration

As the duration of a flood extends, the probability of failure inevitably increases, as extended flooding increases pore pressures, and increases the likelihood and intensity of damaging erosion. The analyses herein essentially assume that the flood has been of sufficient duration that steady-state seepage conditions have developed in pervious substratum materials and pervious embankment materials, but no pore pressure adjustment has occurred in impervious clayey foundation and embankment materials. These are reasonable assumptions for economic analysis of most levees. Further research will be required to provide a rational basis for modifying these functions for flood duration.

Length of Levee and Spatial Correlation

The analyses illustrated herein are for a two-dimensional levee cross section, assumed representative of conditions of a reach of levee extending some unspecified length. Real levees may be a number of miles in length, and reaches are not in fact discrete entities, but rather a continuum. The details of determining the probability of failure for the entire length of levee are beyond the scope of this preliminary report, but several first-cut approximations are noteworthy.

If the levee system were modeled as a series system of discrete independent reaches, such as links in a chain, the reliability is the product of the reliabilities for each link, and the same mathematics holds for combining probabilities as noted above for modes; hence:

$$R = R_1 R_2 R_3 \ldots R_N \tag{40}$$

where the subscripts refer to the separate reaches. Hence the probability of failure for the system is:

$$\begin{aligned} Pr(f) &= 1 - R \\ &= 1 - (1-p_1)(1-p_2)(1-p_3) \ldots (1-p_N) \end{aligned} \tag{41}$$

The problem thus degenerates to that of determining an "equivalent length" of levee for which the soil properties can be taken as statistically independent of adjacent reaches. Much research has been done in the areas of spatial correlation, autocorrelation functions, variance reduction functions, etc., which have a direct bearing on this problem. However, there are seldom sufficient data to quantify such functions.

For practical purposes, pending further research, it seems reasonable to pre-identify levee reaches that are likely to be low in reliability, analyze one or more of these, and base the economic evaluation on the most critical reaches, as a levee system is generally no more reliable than its weakest reach.

12 Summary, Conclusions, and Recommendations

Summary

This research effort and report provided a set of "first-cut" examples of the application of reliability theory to the analysis of several modes of levee performance. Using the capacity-demand model, a conditional probability of failure function can be developed for each performance mode as a function of floodwater elevation. Using elementary reliability theory and assuming an independent series system, a composite conditional-probability-of-failure function can then be calculated that reflects all considered failure modes. The developed methodology is intended to be used as a component in the economic analysis of existing levees.

Conclusions

This effort was the first by the Corps of Engineers to cast the problem of predicted geotechnical performance of existing levees in a probabilistic framework. Full implementation of a probabilistic approach to levee performance prediction will undoubtedly require additional research, additional developmental efforts, and experience-building by practicing engineers in the Corps, and decisions by Corps' policy makers. Nevertheless, a number of conclusions can be drawn from the analyses of the two example problems presented herein:

a. The template method presented in current guidance for estimating existing levee reliability does not explicitly account for the several modes of levee performance (e.g., underseepage) as it does not incorporate information regarding foundation conditions.

b. The probabilistic **capacity-demand model** can be used to develop conditional-probability-of-failure functions for levees as functions of floodwater elevation. In this approach, the probability of failure is taken to be a function of the quantified uncertainty in the engineering parameters used in performance analysis of the levee.

c. For **underseepage** analysis, relatively high probabilities of failure can be present for some commonly encountered foundation conditions. In the probabilistic analysis of the example problems, it was found that the top blanket thickness (z) is the major contributor to the uncertainty in performance. This is consistent with previous analyses of similar problems (Shannon and Wilson, Inc., and Wolff (1994)).

d. For **slope stability** analysis, probabilities of failure calculated for the two example problems were considerably lower than those for seepage analysis. This is also consistent with previous studies on similar problems (Shannon and Wilson, Inc., and Wolff (1994)). In general, floodwater elevation does not significantly affect the probability of slope failure except for pervious levees where through-seepage may induce slope instability.

e. For **through-seepage** analysis, further review of available deterministic analysis models is required. For the example analyses herein, an adaptation of Rock Island procedures was used to illustrate a probabilistic approach. However, the procedure used is based on criteria for conditions at which the construction of berms is recommended, and does not represent a true limit state or condition where erosion may result in levee failure.

f. For **surface erosion**, a conceptual example was presented based on average current velocities determined from a simplified Manning equation and an assumed scour velocity. For actual levees under study, better characterization of the actual current velocity can likely be obtained from existing hydraulic models used by the Corps, and a better characterization of the critical velocity that will induce damaging scour can also likely be developed. Furthermore, the occurrence of damaging scour does not necessarily imply that levee failure will occur, and some adjustment of results may be necessary to account for this.

g. **Surface erosion** can also be induced by wave attack. A similar analytical model should be developed for this condition, which was beyond the scope of the present effort.

h. **Engineering judgment** regarding the probability of failure **for modes other than those analyzed** can be incorporated into the analysis so long as it can be quantified. For example, deficiencies such as cracks or animal burrows observed in a field inspection can be included by having the engineer assign judgmental probability-of-failure functions reflecting observed conditions.

i. As a first approximation, the several conditional probability-of-failure functions for the considered performance modes were combined assuming **independence** of performance modes and functioning as a **series system**. However, there is undoubtedly some correlation between some modes; for example, through-seepage and slope stability, which should be considered in further development of the methodology.

Each analysis presented herein was based on a **single formulation** of the problem (e.g., a defined set of random variables and the performance function used with the Taylor's series method). In order to be in a position to recommend the best specific approaches for application in practice, further research and refinement of these analyses are required to evaluate and compare a number of alternative formulations in the probabilistic methods, the effect of various assumptions, etc.

Incorporation of **length effects** requires further research. The example analyses herein provide the combined probability of failure function for a two-dimensional levee cross section representative of an unspecified length. Sections very close to the analyzed section will be highly correlated with that section, and hence the analyzed section can be considered to model some equivalent "statistically homogeneous" length of levee. Sections at some distance can be considered to represent another equivalent length of levee. The entire levee length can then be considered as a chain, with each equivalent section an independent link. Probabilistic techniques are readily available to analyze such a system once the number and size of links and the distribution of their probabilities of failure are determined; however, much work remains to be done in developing methodology for that specific step.

Recommendations

To continue development and implementation of a probabilistic approach to assessment of existing levees, the following activities are recommended:

a. **Development and revision of software**, to enhance practitioners' capability to fit probability distribution or moments to random variables, and to perform probabilistic seepage and stability analysis.

b. **Additional research**, with examples similar to those herein, **on a wider range of levee conditions** and considering **and testing possible alternative approaches** in characterizing variables, defining performance functions, calculating probabilistic moments, etc.

c. **Additional research on length effects and spatial correlation** effects, as previously described.

d. **Initial research on the probabilistic frequency and categorization of levee performance** problems, to begin calibration of developed procedures against observed performance.

e. **Training** of geotechnical engineers expected to use the developed methodology.

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Annex A
Brief Review of Probability and Reliability Terms and Concepts

Introduction

The objective of this annex is to introduce some basis elements of engineering reliability analysis applicable to geotechnical structures for various modes of performance. These reliability measures are intended to be sufficiently consistent and suitable for application to economic analysis of geotechnical structures of water resource projects. References are provided which should be consulted for detailed discussion of the principles of reliability analyses.

Traditionally, evaluations of geotechnical adequacy are expressed by safety factors. A safety factor can be expressed as the ratio of capacity to demand. The safety concept, however, has shortcomings as a measure of the relative reliability of geotechnical structures for different performance modes. A primary deficiency is that parameters (material properties, strengths, loads, etc.) must be assigned single, precise values when the appropriate values may in fact be uncertain. The use of precisely defined single values in an analysis is known as the *deterministic* approach. The safety factor using this approach reflects the condition of the feature, the engineer's judgement, and the degree of conservatism incorporated into the parameter values.

Another approach, the *probabilistic* approach, extends the safety factor concept to explicitly incorporate uncertainty in the parameters. This uncertainty can be quantified through statistical analysis of existing data or judgementally assigned. Even if judgementally assigned, the probabilistic results will be more meaningful than a deterministic analysis because the engineer provides a measure of the uncertainty of his or her judgement in each parameter.

Reliability Analysis Principles

The probability of failure

Engineering reliability analysis is concerned with finding the *reliability* R or the *probability of failure* $Pr(f)$ of a feature, structure, or system. As a system is considered reliable unless it fails, the reliability and probability of failure sum to unity:

$$R + Pr(f) = 1 \tag{A1}$$

$$R = 1 - Pr(f) \tag{A2}$$

$$Pr(f) = 1 - R \tag{A3}$$

In the engineering reliability literature, the term *failure* is used to refer to any occurrence of an adverse event under consideration, including simple events such as maintenance items. To distinguish adverse but noncatastrophic events (which may require repairs and associated expenditures) from events of catastrophic failure (as used in the dam safety context), the term *probability of unsatisfactory performance* $Pr(U)$ is sometimes used. An example would be slope stability where the safety factor is below the required minimum safety factor but above 1.0. Thus for this case, reliability is defined as:

$$R = 1 - Pr(U) \tag{A4}$$

Contexts of reliability analysis

Engineering reliability analysis can be used in several general contexts:

a. Estimation of the reliability of a new structure or system upon its construction and first loading.

b. Estimation of the reliability of an existing structure or system upon a new loading.

c. Estimation of the probability of a part or system surviving for a given lifetime.

Note that the third context has an associated time interval, whereas the first two involve measures of the overall adequacy of the system in response to a load event.

Reliability for the first two contexts can be calculated using the *capacity-demand model* and quantified by the *reliability index* β. In the capacity-demand model, uncertainty in the performance of the structure or system is taken to be a function of the uncertainty in the values of various parameters used in calculating some measure of performance, such as the factor of safety.

In the third context, reliability over a future time interval is calculated using parameters developed from actual data on the lifetimes or frequencies of failure of similar parts or systems. These are usually taken to follow the exponential or Weibull probability distributions. This methodology is well-established in electrical, mechanical, and aerospace engineering, where parts and components routinely require periodic replacement. This approach produces a *hazard function* which defines the probability of failure in any time period. These functions are used in economic analysis of proposed geotechnical improvements.

For reliability evaluation of most geotechnical structures, in particular existing levees, the capacity-demand model will be utilized, as the question of interest is the probability of failure related to a load event rather than the probability of failure within a time interval.

Reliability Index

The reliability index β is a measure of the reliability of an engineering system that reflects both the mechanics of the problem and the uncertainty in the input variables. This index was developed by the structural engineering profession to provide a measure of comparative reliability without having to assume or determine the shape of the probability distribution necessary to calculate an exact value of the probability of failure. The reliability index is defined in terms of the expected value and standard deviation of the performance function, and permits comparison of reliability among different structures or modes of performance without having to calculate absolute probability values. Calculating the reliability index requires:

a. A deterministic model (e.g., a slope stability analysis procedure).

b. A performance function (e.g., the factor of safety from UTEXAS2).

c. The expected values and standard deviations of the parameters taken as random variables (e.g., $E[\phi]$ and σ_{ϕ}).

d. A definition of the limit state (e.g., $ln(FS) = 0$).

e. A method to estimate the expected value and standard deviation of the limit state given the expected values and standard deviations of the parameters (e.g., the Taylor's series or point estimate methods).

Accuracy of Reliability Index

For rehabilitation studies of geotechnical structures, the reliability index is used as a “relative measure of reliability or confidence in the ability of a structure to perform its function in a satisfactory manner.”

The analysis methods used to calculate the reliability index should be sufficiently accurate to rank the relative reliability of various structures and components. However, reliability index values are not absolute measures of probability. Structures, components, and performance modes with higher indices are considered more reliable than those with lower indices. Experience analyzing geotechnical structures will refine these techniques.

The Capacity-Demand Model

In the capacity-demand model, the probability of failure or unsatisfactory performance is defined as the probability that the demand on a system or component exceeds the capacity of the system or component. The capacity and demand can be combined into a single function (*the performance function)*, and the event that the capacity equals the demand taken as the *limit state*. *Reliability R* is the probability that the limit state will not be achieved or crossed.

The concept of the capacity-demand model is illustrated for slope stability analysis in Figure A1. Using the expected value and standard deviation of the random variables c and ϕ in conjunction with the Taylor's series method or the point estimate method, the expected value and standard deviation of the factor of safety can be calculated. If it is assumed that the factor of safety is lognormally distributed, then the natural log of the factor of safety is normally distributed.

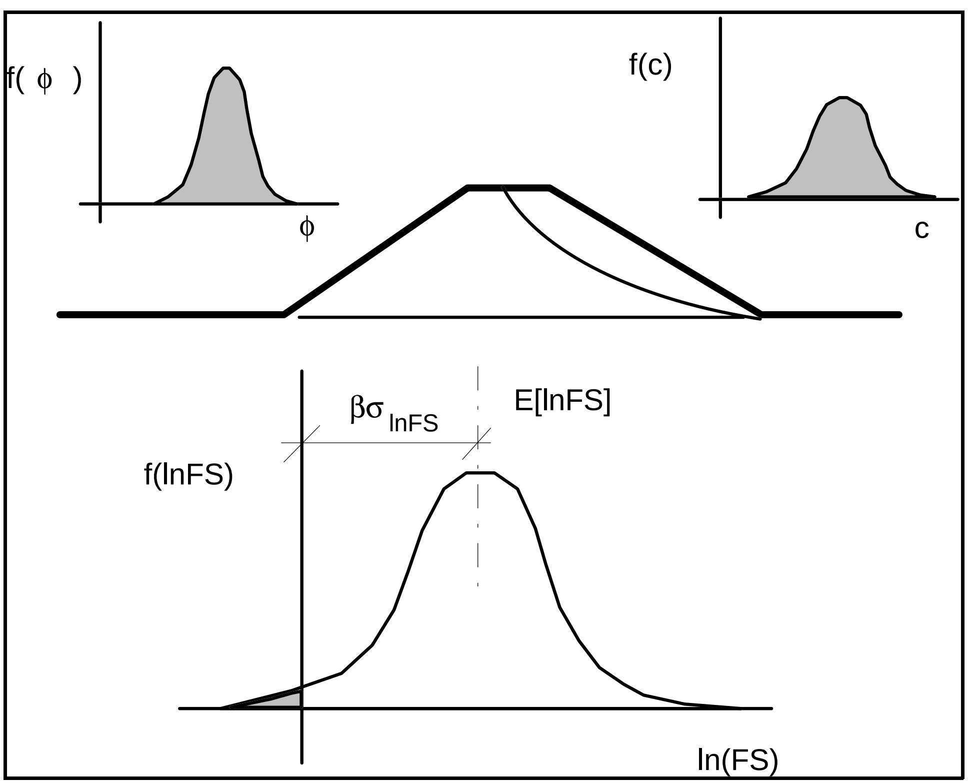

Figure A1. The capacity-demand model

The performance function is taken as the log of the factor of safety, and the limit state is taken as the condition $ln(FS) = 0$. The probability of failure is then the shaded area corresponding to the condition $ln(FS) < 0$. If it is assumed that the distribution on $ln(FS)$ is normal, then the probability of failure can be obtained using standard statistical tables.

Equivalent performance functions and limit states can be defined using other measures, such as the exit gradient for seepage.

The probability of failure associated with the reliability index is a *probability per structure*; it has no time-frequency basis. Once a structure is constructed or loaded as modeled, it either performs satisfactorily or not. Nevertheless, the β value calculated for an existing structure provides a rational comparative measure.

Steps in a Reliability Analysis Using the Capacity-Demand Model

As suggested by Figure A1 for slope stability, a reliability analysis includes the following steps:

a. Important variables considered to have sufficient inherent uncertainty are taken as random variables and characterized by their expected values, standard deviations, and correlation coefficients. In concept, every variable in an analysis can be modeled as a random variable as most properties and parameters have some inherent variability and uncertainty. However, a few specific random variables will usually dominate the analysis. Including additional random variables may unnecessarily increase computational effort without significantly improving results. When in doubt, a few analyses with and without certain random variables will quickly illustrate which are significant, as will the examination of variance terms in a Taylor's series analysis. For levee analysis, significant random variables typically include material strengths, soil permeability or permeability ratio, and thickness of top stratum. Material properties such as soil density may be significant, but where strength and density both appear in an analysis, strength may dominate. An example of a variable that can be represented deterministically (non-random) is the density of water.

b. A performance function and limit state are identified.

c. The expected value and standard deviation of the performance function are calculated. In concept, this involves integrating the performance function over the probability density functions of the random variables. In practice, approximate values are obtained using the expected value, standard deviation, and correlation coefficients of the random variables in the Taylor's series method or the point estimate method.

d. The reliability index β is calculated from the expected and standard deviation of the performance function. The reliability index is a measure of the distance between the expected value of ln (*C/D*) or ln (*FS*) and the limit state.

e. If a probability of failure value is desired, a distribution is assumed and *Pr*(*f*) is calculated.

Random Variables

Description

Parameters having significance in the analysis and some significant uncertainty are taken as *random variables.* Instead of having precise single values, random variables assume a range of values in accordance with a *probability density function* or *probability distribution*. The probability distribution quantifies the likelihood that its value lies in any given interval. Two commonly used distributions, the normal and the lognormal, are described later in this appendix.

Moments of random variables

To model random variables in the Taylor's series or point estimate methods, one must provide their expected values and standard deviations, which are two of several probabilistic *moments* of a random variable. These can be calculated from data or estimated from experience. For random variables which are not independent of each other, but tend to vary together, correlation coefficients must also be assigned.

Mean value. The *mean* value μ_x of a set of N measured values for the random variable X is obtained by summing the values and dividing by N:

$$\mu_x = \frac{\sum_{i=1}^{N} X_i}{N} \tag{A5}$$

Expected value. The *expected value* $E[X]$ of a random variable is the mean value one would obtain if all possible values of the random variable were multiplied by their likelihood of occurrence and summed. Where a mean value can be calculated from representative data, it provides an unbiased estimate of the expected value of a parameter; hence, the mean and expected value are numerically the same. The expected value is defined as:

$$E[X] = \mu_X = \int X f(X) dx \approx \sum X p(X_i) \tag{A6}$$

where $f(X)$ is the *probability density function* of X (for continuous random variables) and $p(X_i)$ is the probability of the value X_i (for discrete random variables).

Variance. The *variance Var*[X] of a random variable X is the expected value of the squared difference between the random variable and its mean value. Where actual data are available, the variance of the data can be calculated by subtracting each value from the mean, squaring the result, and determining the average of these values:

$$Var[X] = E[(X-\mu_X)^2] = \int (X-\mu_X)^2 f(X)dX = \frac{\sum [(X_i-\mu_X)^2]}{N} \qquad \text{(A7)}$$

The summation form above involving the X_i term provides the variance of a population containing exactly N elements. Usually, a *sample* of size N is used to obtain an *estimate of the variance* of the associated random variable which represents an *entire population* of items or continuum of material. To obtain an unbiased estimate of the population working from a finite sample, the N is replaced by *N-1*:

$$Var[X] = \frac{\sum [(X_i-\mu_X)^2]}{N-1} \qquad \text{(A8)}$$

Standard deviation. To express the scatter or dispersion of a random variable about its expected value in the same units as the random variable itself, the *standard deviation* is taken as the square root of the variance; thus:

$$\sigma_X = \sqrt{Var[X]} \qquad \text{(A9)}$$

Coefficient of variation. To provide a convenient dimensionless expression of the uncertainty inherent in a random variable, the standard deviation is divided by the expected value to obtain the *coefficient of variation*, which is usually expressed as a percent:

$$V_X = \frac{\sigma_X}{E[X]} \times 100\% \qquad \text{(A10)}$$

The expected value, standard deviation, and coefficient of variation are inter-dependent: knowing any two, the third is known. In practice, a convenient way to estimate moments for parameters where little data are available is to assume that the coefficient of variation is similar to previously measured values from other data sets for the same parameter.

Correlation

Pairs of random variables may be correlated or independent; if correlated, the likelihood of a certain value of the random variable Y depends on the value of the random variable X. For example, the strength of sand may be correlated with density or the top blanket permeability may be correlated with grain size of the sand. The *covariance* is analogous to the variance but measures the combined effect of how two variables vary together. The definition of the covariance is:

$$Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)] \tag{A11}$$

which is equivalent to:

$$Cov[X,Y] = \int\int(X-\mu_X)(Y-\mu_Y)f(X,Y)dYdX \tag{A12}$$

In the above equation, $f(X, Y)$ is the joint probability density function of the random variables X and Y. To calculate the covariance from data, the following equation can be used:

$$Cov[X,Y] = \frac{1}{N}\sum(X_i-\mu_X)(Y_i-\mu_Y) \tag{A13}$$

To provide a nondimensional measure of the degree of correlation between X and Y, the *correlation coefficient* $\rho_{X,Y}$, is obtained by dividing the covariance by the product of the standard deviations:

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sigma_X\sigma_Y} \tag{A14}$$

The correlation coefficient may assume values from -1.0 to +1.0. A value of 1.0 or -1.0 indicates there is perfect linear correlation; given a value of X, the value of Y is known and hence is not random. A value of zero indicates no linear correlation between variables. A positive value indicates the variables increase and decrease together; a negative value indicates that one variable decreases as the other increases. Pairs of *independent* random variables have zero correlation coefficients.

Probability Distributions

Definition

The terms *probability distribution*, *probability density function*, *pdf*, or the notation $f_X(X)$ refer to a function that defines a continuous random variable. The Taylor's series and point estimate methods described herein to determine moments of performance functions require only the mean and standard deviation of random variables and their correlation coefficients; knowledge of the form of the probability density function is not necessary. However, in order to ensure that estimates made for these moments are reasonable, it is recommended that the engineer plot the shape of the normal or lognormal distribution which has the expected value and standard deviation assumed. This can easily be done with spreadsheet software.

Figure A1 illustrated probability density functions for the random variables c and ϕ. A probability density function has the property that for any X, the value of $f(x)$ is proportional to the likelihood of X. The area under a probability density function is unity. The probability that the random variable X lies between two values X_1 and X_2 is the integral of the probability density function taken between the two values. Hence:

$$Pr\,(X_1 < X < X_2) = \int_{X_1}^{X_2} f_X(X)dx \tag{A15}$$

The *cumulative distribution function CDF* or $F_X(X)$ measures the integral of the probability density function from minus infinity to X:

$$F_X(X) = \int_{-\infty}^{X} f_X(X)dx \tag{A16}$$

Thus, for any value X, $F_X(X)$ is the probability that the random variable X is less than the given x.

Estimating Probabilistic Distributions

A suggested method to assign or check assumed moments for random variables is to:

a. Assume trial values for the expected value and standard deviation and take the random variable to be normal or lognormal.

b. Plot the resulting density function and tabulate and plot the resulting cumulative distribution function (spreadsheet software is a convenient way to do this).

c. Assess the reasonableness of the shape of the pdf and the values of the CDF.

d. Repeat the above steps with successively improved estimates of the expected value and standard deviation until an appropriate pdf and CDF are obtained.

Normal distribution

The *normal* or *Gaussian* distribution is the most well-known and widely assumed probability density function. It is defined in terms of the mean μ_X and standard deviation σ_X as:

$$f_X(X) = \frac{1}{\sigma_x\sqrt{2\pi}}\exp\left[-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right] \tag{A17}$$

When fitting the normal distribution, the mean of the distribution is taken as the expected value of the random variable. The cumulative distribution function for the normal distribution is not conveniently expressed in closed form but is widely tabulated and can be readily computed by numerical approximation. It is a built-in function in most spreadsheet programs. Although the normal distribution has limits of plus and minus infinity, values more than 3 or 4 standard deviations from the mean have very low probability. Hence, one empirical fitting method is to take minimum and maximum reasonable values to be at approximately ±3 standard deviations. The normal distribution is commonly assumed to characterize many random variables where the coefficient of variation is less than about 30 percent. For levees, these include soil density and drained friction angle. Where the mean and standard deviation are the only information known, it can be shown that the normal distribution is the most unbiased choice.

Lognormal distribution

When a random variable X is lognormally distributed, its natural logarithm, $ln\ X$, is normally distributed. The lognormal distribution has several properties which often favor its selection to model certain random variables in engineering analysis:

a. As X is positive for any value of $ln\ X$, lognormally distributed random variables cannot assume values below zero.

b. It often provides a reasonable shape in cases where the coefficient of variation is large (>30 percent) or the random variable may assume values over one or more orders of magnitude.

c. The central limit theorem implies that the distribution of products or ratios of random variables approaches the lognormal distribution as the number of random variables increases.

If the random variable X is lognormally distributed, then the random variable $Y = ln\ X$ is normally distributed with parameters $E[Y] = E[ln\ X]$ and $\sigma_Y = \sigma_{lnX}$. To obtain the parameters of the normal random variable Y, first the coefficient of variation of X is calculated:

$$V_X = \frac{\sigma_X}{E[X]} \tag{A18}$$

The standard deviation of *Y* is then calculated as:

$$\sigma_Y = \sigma_{\ln X} = \sqrt{\ln(1+V_X^2)} \tag{A19}$$

The standard deviation σ_Y is in turn used to calculate the expected value of *Y*:

$$E[Y] = E[lnX] = lnE[X] - \frac{\sigma_Y^2}{2} \tag{A20}$$

The density function of the lognormal variate *X* is:

$$f(X) = \frac{1}{X\sigma_Y\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln X - E[Y]}{\sigma_Y}\right)^2\right] \tag{A21}$$

The shape of the distribution can be plotted from the above equation. Values on the cumulative distribution function for *X* can be determined from the cumulative distribution function of *Y* ($E[Y]$, σ_Y) by substituting the *X* in the expression $Y = ln\ X$.

Calculation of the Reliability Index

As illustrated in Figure A2, a simple definition of the reliability index is based on the assumption that capacity and demand are normally distributed and the limit state is the event that their difference, the safety margin *S*, is zero. The random variable *S* is then also normally distributed and the reliability index is the distance by which $E[S]$ exceeds zero in units of σ_S:

$$\beta = \frac{E[S]}{\sigma_S} = \frac{E[C-D]}{\sqrt{\sigma_C^2+\sigma_D^2}} \tag{A22}$$

An alternative formulation (also shown in Figure A2) implies that capacity *C* and demand *D* are lognormally distributed random variables. In this case, *ln C* and *ln D* are normally distributed. Defining the factor of safety *FS* as the ratio *C*/*D*, then *ln FS* = (*ln C*) - (*ln D*) and *ln FS* is normally distributed. Defining the reliability index as the distance by which *ln FS* exceeds zero in terms of the standard deviation of *ln FS*, it is:

$$\beta = \frac{E[lnC - lnD]}{\sigma_{(lnC-lnD)}} = \frac{E[\ln(C/D)]}{\sigma_{\ln(C/D)}} = \frac{E[lnFS]}{\sigma_{lnFS}} \tag{A23}$$

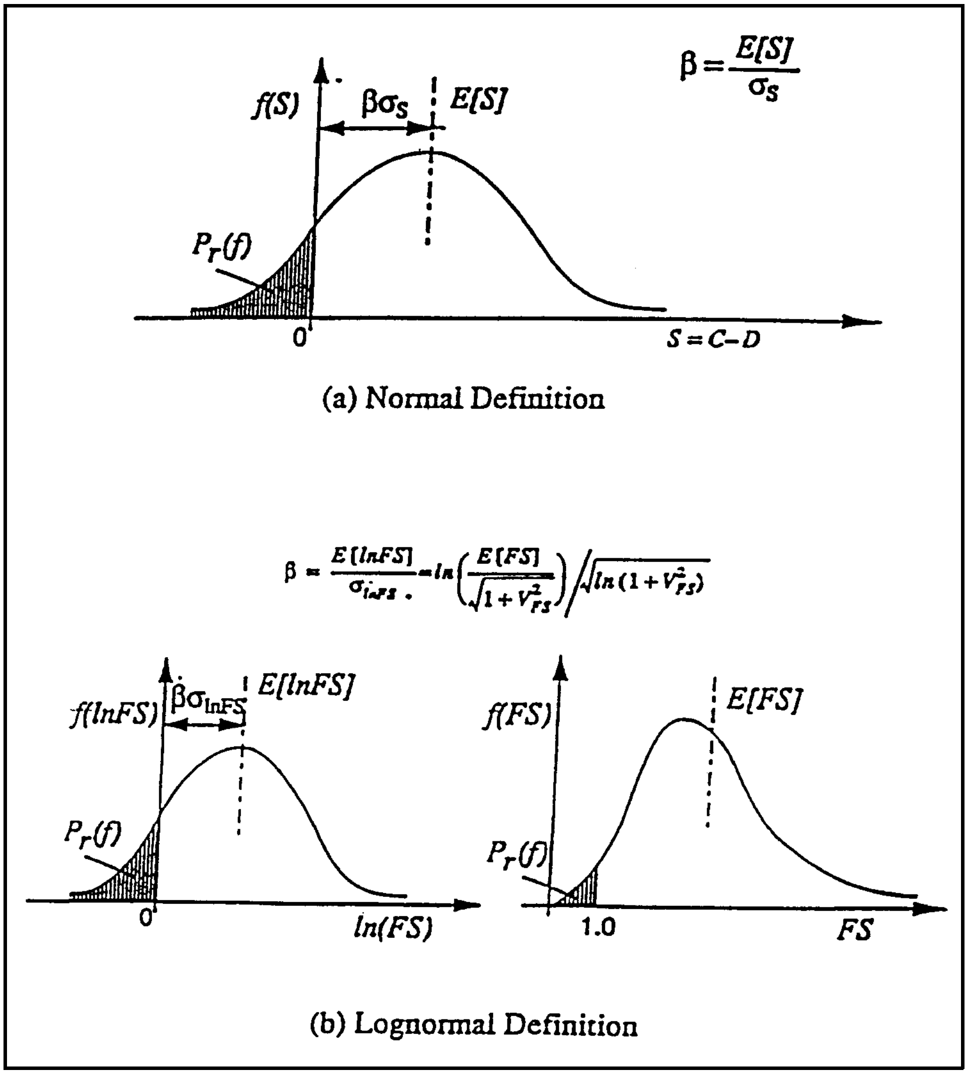

Figure A2. Alternative definitions of the reliability index

From the properties of the lognormal distribution, the expected value of *ln C* is:

$$E[lnC] = lnE[C] - \frac{1}{2}\sigma^2_{lnC} \quad \text{(A24)}$$

where:

$$\sigma^2_{\ln C} = \ln\left[1 + V_C^2\right] \quad \text{(A25)}$$

Similar expressions apply to $E[lnD]$ and σ_{lnD}.

The expected value of the log of the factor of safety is then:

$$E[\ln FS] = \ln E[C] - lnE[D] - \frac{1}{2}\ln\left[1+V_C^2\right] + \frac{1}{2}\ln\left[1+V_D^2\right] \quad \text{(A26)}$$

As the second-order terms are small when the coefficients of variation are not exceedingly large (below approximately 30 percent), the equation above is sometimes approximated as:

$$E[lnFS] \approx lnE[C] - lnE[D] = \ln\left[\frac{E[C]}{E[D]}\right] \quad \text{(A27)}$$

The standard deviation of the log of the factor of safety is obtained as:

$$\sigma_{lnFS} = \sqrt{\sigma_{lnC}^2 + \sigma_{lnD}^2} \quad \text{(A28)}$$

$$\sigma_{\ln FS} = \sqrt{\ln\left[1+V_C^2\right] + \ln\left[1+V_D^2\right]} \quad \text{(A29)}$$

Introducing an approximation,

$$\ln\left[1+V_C^2\right] \approx V_C^2 \quad \text{(A30)}$$

the reliability index for lognormally distributed *C*, *D*, and *FS* and normally distributed *ln C*, *ln D*, and *ln FS* can be expressed approximately as:

$$\beta = \frac{\ln\left(\frac{E[C]}{E[D]}\right)}{\sqrt{V_C^2 + V_D^2}} \quad \text{(A31)}$$

The exact expression is:

$$\beta = \frac{\ln\left[\frac{E[C]\sqrt{1+V_D^2}}{E[D]\sqrt{1+V_C^2}}\right]}{\sqrt{\ln[1+v_C^2] \ + \ \ln[1+V_D^2]}} \quad \text{(A32)}$$

For many geotechnical problems and related deterministic computer programs, the output is in the form of the factor of safety, and the capacity and demand are not explicitly separated. The reliability index must be calculated from values of $E[FS]$ and σ_{FS} obtained from multiple runs as later described in the next section. In this case, the reliability index is obtained using the following steps:

$$V_{FS} = \frac{\sigma_{FS}}{E[FS]} \quad \text{(A33)}$$

$$\sigma_{\ln FS} = \sqrt{\ln(1+V_{FS}^2)} \tag{A34}$$

$$E[\ln FS] = \ln E[FS] - \frac{1}{2}\ln(1+V_{FS}^2) \tag{A35}$$

$$\beta = \frac{E[\ln FS]}{\sigma_{\ln FS}} = \frac{\ln\left[E[FS]/\sqrt{1+V_{FS}^2}\right]}{\sqrt{\ln\left(1+V_{FS}^2\right)}} \tag{A36}$$

Integration of the Performance Function

Methods such as direct integration, Taylor's series, point estimate methods, and Monte Carlo simulation are available for calculating the mean and standard deviation of the performance function. For direct integration, the mean value of the function is obtained by integrating over the probability density function of the random variables. A brief description of the other methods follows. The References section that follows the main text of this report should be consulted for additional information.

The Taylor's series method

The Taylor's series method is one of several methods to estimate the moments of a performance function based on moments of the input random variables [see Harr (1987)]. It is based on a Taylor's series expansion of the performance function about some point. For the Corps' navigation rehabilitation studies, the expansion is performed about the expected values of the random variables. The Taylor's series method is termed a first-order, second-moment (FOSM) method, as only first-order (linear) terms of the series are retained and only the first two moments (mean and the standard deviation) are considered. The method is summarized below and illustrated by an example in Annex B.

Independent random variables. Given a function $Y = g(X_1, X_2, \ldots X_n)$, where all X_i are independent, the expected value of the function is obtained by evaluating the function at the expected values of the random variables:

$$E[Y] = g\left(E[X_1], E[X_2], \ldots E[X_n]\right) \tag{A37}$$

For a function such as the factor of safety, this implies that the expected value of the factor of safety is calculated using the expected values of the random variables:

$$E[FS] = FS\left(E[\phi_1], E[c_1], E[\gamma_1], \ldots\right) \tag{A38}$$

The variance of the performance function is taken as:

$$Var[Y] = \sum\left[\left(\frac{\partial Y}{\partial X_i}\right)^2 VarX_i\right] \tag{A39}$$

with the partial derivatives taken at the expansion point (in this case the mean or expected value). Using the factor of safety as an example performance function, the variance is obtained by finding the partial derivative of the factor of safety with respect to each random variable evaluated at the expected value of that variable, squaring it, multiplying it by the variance of that random variable, and summing these terms over all of the random variables:

$$Var[FS] = \sum\left[\left(\frac{\partial FS}{\partial X_i}\right)^2 VarX_i\right] \tag{A40}$$

The standard deviation of the factor of safety is then simply the square root of the variance.

Having the expected value and variance of the factor of safety, the reliability index can be calculated as described earlier in this annex. Advantages of the Taylor's Series method include the following:

a. The relative magnitudes of the terms in the above summation provide an explicit indication of the relative contribution of uncertainty of each variable.

b. The method is exact for linear performance functions.

Disadvantages of the Taylor's Series method include the following:

a. It is necessary to determine the value of derivatives.

b. The neglect of higher-order terms introduces errors for nonlinear functions.

The required derivatives can be estimated numerically by evaluating the performance function at two points. The function is evaluated at one increment above and below the expected value of the random variable X_i and the difference of the results is divided by the difference between the two values of X_i. Although the derivative at a point is most precisely evaluated using a very small increment, evaluating the derivative over a range of ±1 standard deviation may better capture some of the nonlinear behavior of the function over a range of likely values. Thus, the derivative is evaluated using the following approximation:

$$\frac{\partial Y}{\partial X_i} = \frac{g\left(E\left[X_i\right] + \sigma_{X_i}\right) - g\left(E\left[X_i\right] - \sigma_{X_i}\right)}{2\sigma_{X_i}} \tag{A41}$$

When the above expression is squared and multiplied by the variance, the standard deviation term in the denominator cancels the variance, leading to

$$\left(\frac{\partial Y}{\partial X_i}\right)^2 VarX = \left[\frac{g\left(X_+\right) - g\left(X_-\right)}{2}\right]^2 \tag{A42}$$

where X_+ and X_- are values of the random variable at plus and minus one standard deviation from the expected value.

Correlated random variables. Where random variables are correlated, solution is more complex. The expression for the expected value, retaining second-order terms is:

$$E[Y] = g\left(E[X_1], E[X_2], \ldots E[X_n] + \frac{1}{2}\sum \frac{\partial^2 Y}{\partial X_i \partial X_j} Cov\left(X_i X_j\right)\right) \tag{A43}$$

However, in keeping with the first-order approach, the second-order terms are generally neglected, and the expected value is calculated the same as for independent random variables.

The variance, however, is taken as:

$$Var[Y] = \sum\left[\left(\frac{\partial Y}{\partial X_i}\right)^2 VarX_i\right] + 2\sum\left[\frac{\partial Y}{\partial X_i}\frac{\partial Y}{\partial X_j} Cov\left(X_i X_j\right)\right] \tag{A44}$$

where the covariance part contains terms for each possible combination of random variables.

The Point Estimate Method

An alternative method to estimate moments of a performance function based on moments of the random variables is the *point estimate method*. Point estimate methods are procedures where probability distributions for continuous random variables are modeled by discrete "equivalent" distributions having two or more values. The elements of these discrete distributions (or *point estimates*) have specific values with defined probabilities such that the first few moments of the discrete distribution match that of the continuous random variable. Having only a few values over which to integrate, the moments of the performance function are easily obtained. A simple and straightforward point estimate method has been proposed by Rosenblueth (1975, 1981) and is summarized by Harr (1987). That method is briefly summarized below and illustrated by example in Annex B.

Independent random variables

As shown in Figure A3, a continuous random variable X is represented by two point estimates, X_+ and X_-, with probability concentrations P_+ and P_-, respectively. As the two point estimates and their probability concentrations form an equivalent probability distribution for the random variable, the two P values must sum to unity. The two point estimates and probability concentrations are chosen to match three moments of the random variable. When these conditions are satisfied for symmetrically distributed random variables, the point estimates are taken at the mean ±1 standard deviation:

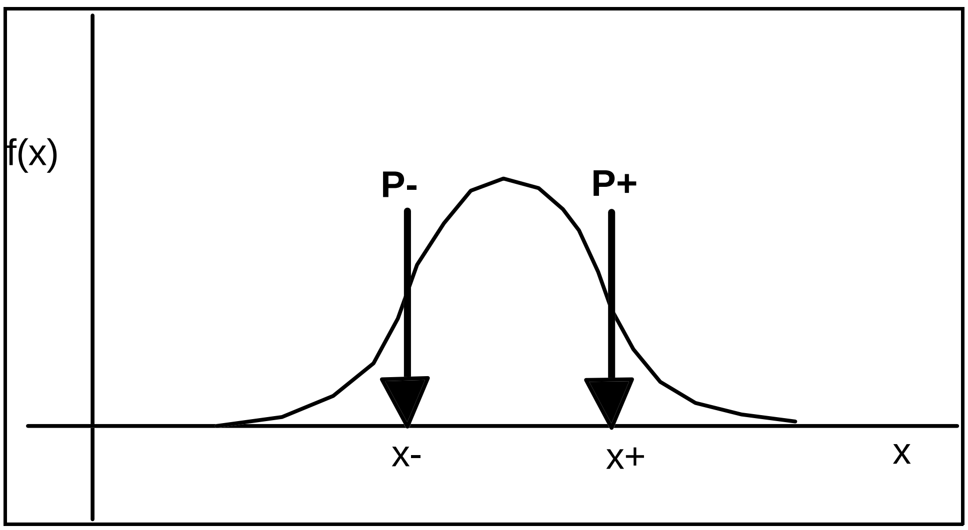

Figure A3. Point estimate method

$$X_{i+} = E[X_i] + \sigma_{Xi} \quad \text{(A45)}$$

$$X_{i-} = E[X_i] - \sigma_{Xi} \quad \text{(A46)}$$

For independent random variables, the associated probability concentrations are each one-half:

$$P_{i+} = P_{i-} = 0.50 \quad \text{(A47)}$$

Knowing the point estimates and their probability concentrations for each variable, the expected value of a function of the random variables raised to any power M can be approximated by evaluating the function for each possible combination of the point estimates (e.g., X_{1+}, X_{2-}, X_{3+}, X_{n-}), multiplying each result by the product of the associated probability concentrations (e.g., $P_{+--} = P_{1+}P_{2-}P_{3-}$) and summing the terms. For example, two random variables result in four combinations of point estimates and four terms:

$$E[Y^M] = P_{++}g(X_{1+}, X_{2+})^M + P_{+-}g(X_{1+}, X_{2-})^M + P_{-+}g(X_{1-}, X_{2+})^M + P_{--}g(X_{1-}, X_{2-})^M \qquad \text{(A48)}$$

For N random variables, there are 2^N combinations of the point estimates and 2^N terms in the summation. To obtain the expected value of the performance function, the function $g(X_1, X_2)$ is calculated 2^N times using all the combinations and the exponent M in Equation A48 is 1. To obtain the standard deviation of the performance function, the exponent M is taken as 2 and the squares of the obtained results are weighted and summed to obtain $E[Y^2]$. The variance can then be obtained from the identity

$$Var[Y] = E[Y^2] - (E[Y])^2 \qquad \text{(A49)}$$

and the standard deviation is the square root of the variance.

Correlated random variables

Correlation between symmetrically distributed random variables is treated by adjusting the probability concentrations ($P_{\pm\pm....\pm}$). A detailed discussion is provided by Rosenblueth (1975) and summarized by Harr (1987). For certain geotechnical analyses involving lateral earth pressure, bearing capacity of shallow foundations, and slope stability, often only two random variables (c and ϕ or $\tan \phi$) need to considered as correlated. For two correlated random variables within a group of two or more, the product of their concentrations is modified by adding a correlation term:

$$P_{i+j-} = P_{i-j+} = (P_{i-})(P_{j+}) - \frac{\rho}{4} \qquad \text{(A50)}$$

$$P_{i+j+} = P_{i-j-} = (P_{i+})(P_{j+}) + \frac{\rho}{4} \qquad \text{(A51)}$$

Monte Carlo simulation

The performance function is evaluated for many possible values of the random variables. A plot of the results will produce an approximation of the probability distribution. Once the probability distribution is determined in this manner, the mean and standard deviation of the distribution can be calculated.

Determining the Probability of Failure

Once the expected value and standard deviation of the performance function have been determined using the Taylor's Series or point estimate methods, the reliability index can be calculated as previously described. If the reliability index is assumed to be the number of standard deviations by which the expected value of a normally distributed performance function (e.g., $ln(FS)$) exceeds zero, then the probability of failure can be calculated as:

$$Pr(f) = \Psi(-\beta) = \Psi(-z) \tag{A52}$$

where $\psi(-z)$ is the cumulative distribution function of the standard normal distribution evaluated at $-z$, which is widely tabulated and available as a built-in function on modern microcomputer spreadsheet programs.

Overall System Reliability

Reliability indices for a number of components or a number of modes of performance may be used to estimate the overall reliability of an embankment. There are two types of systems that bound the possible cases, the series system and the parallel system.

Series system

In a series system, the system will perform unsatisfactorily if any one component performs unsatisfactorily. If a system has n components in series, the probability of unsatisfactory performance of the ith component is p_i and its reliability, $R_i = 1 - p_i$, then the reliability of the system, or probability that all components will perform satisfactorily, is the product of the component reliabilities.

$$\begin{aligned} R &= R_1 R_2 R_3 \ldots R_i \ldots R_n \\ &= (1-p_1)(1-p_2)(1-p_3)\ldots(1-p_i)\ldots(1-p_n) \end{aligned} \tag{A53}$$

Simple parallel system

In a parallel system, the system will only perform unsatisfactorily if all components perform unsatisfactorily. Thus, the reliability is unity minus the probability that all components perform unsatisfactorily, or

$$R = 1 - p_1 p_2 p_3 \ldots p_i \ldots p_n \tag{A54}$$

Parallel series systems

Solutions are available for systems requiring *r*-out-of-*n* operable components, which may be applicable to problems such as dewatering with multiple pumps, where *r* is defined as the number of reliable units. Subsystems involving independent parallel and series systems can be mathematically combined by standard techniques.

Upper and lower bounds on system reliability can be determined by considering all components to be from subgroups of parallel and series systems, respectively; however, the resulting bounds may be so broad as to be impractical. A number of procedures are found in the references to narrow the bounds.

Engineering systems such as embankments are complex and have many performance modes. Some of these modes may not be independent; for instance several performance modes may be correlated to the occurrence of a high or low pool level. Rational estimation of the overall reliability of an embankment is a topic that is beyond the scope of this report.

A practical approach

The reliability of a few subsystems or components may govern the reliability of the entire system. Thus, developing a means to characterize and compare the reliability of these components as a function of time is sufficient to make engineering judgements to prioritize operations and maintenance expenditures.

For initial use in reliability assessment of geotechnical systems, the target reliability values presented in the following section should be used. The objective of a rehabilitation program would be to keep the reliability index for each significant mode above the target value for the foreseeable future.

Target Reliability Indices

Reliability indices are a relative measure of the current condition and provide a qualitative estimate of the expected performance. Embankments with relatively high reliability indices will be expected to perform their function well. Embankments with low reliability indices will be expected to perform poorly and present major rehabilitation problems. If the reliability indices are very low, the embankment may be classified as a hazard. The target reliability values shown in Table A1 should be used in general.

Table A1
Target Reliability Indices

Expected Performance Level	Beta	Probability of Unsatisfactory Performance
High	5	0.0000003
Good	4	0.00003
Above average	3	0.001
Below average	2.5	0.006
Poor	2.0	0.023
Unsatisfactory	1.5	0.07
Hazardous	1.0	0.16

Note: Probability of unsatisfactory performance is the probability that the value of performance function will approach the limit state, or that an unsatisfactory event will occur. For example, if the performance function is defined in terms of slope instability, and the probability of unsatisfactory performance is 0.023, then 23 of every 1,000 instabilities will result in damage which causes a safety hazard.

Annex B
Example Calculations of Functions of Random Variables

In this annex, example calculations are provided for three approaches for defining the expected value and standard deviation of a function given the expected values and standard deviations of the input variables.

Problem Statement

The example function considered is the permeability ratio k_f / k_b used in levee underseepage analysis. Note that it could just as well be a performance function, such as the factor of safety in a slope stability analysis. For simplicity of notation, let the permeability ratio be denoted as *PR*; thus:

$$PR = \frac{k_f}{k_b} \qquad \text{(B1)}$$

where k_f is the horizontal permeability of the pervious substratum, and k_b is the vertical permeability of the semipervious top stratum.

Given the following:

$$E[k_f] = 1000\,x\,10^{-4}\,cm/\text{sec} \qquad E[k_b] = 1\,x\,10^{-4}\,cm/\text{sec} \qquad \text{(B2a)}$$

$$\sigma_{kf} = 300\,x\,10^{-4}\,cm/\text{sec} \qquad \sigma_{kb} = 0.3\,x\,10^{-4}\,cm/\text{sec} \qquad \text{(B2b)}$$

$$V_{kf} = V_{kb} = 30\% \qquad \text{(B2c)}$$

It is desired to estimate $E[PR]$, σ_{PR}, and V_{PR}.

Taylor's Series with Exact Derivatives

The expected value of the function, retaining only first-order terms, is the function of the expected values:

$$E[PR] = PR(E[k_F], E[k_b]) = \frac{1000\,x\,10^{-4}}{1\,x\,10^{-4}} = 1000 \tag{B3}$$

As the derivatives of the function are easily obtained, the exact derivatives can be used to calculate the variance. The variance of the permeability ratio is:

$$Var[PR] = \left(\frac{\partial PR}{\partial k_f}\right)^2 \sigma_{kf}^2 + \left(\frac{\partial PR}{\partial k_b}\right)^2 \sigma_{kb}^2 \tag{B4a}$$

$$Var[PR] = \left(\frac{1}{k_b}\right)^2 \sigma_{kf}^2 + \left(\frac{k_f}{-k_b^2}\right) \sigma_{kb}^2 \tag{B4b}$$

The derivatives are evaluated at the expected values of the random variables, giving:

$$Var[PR] = \left(\frac{1}{10^{-4}}\right)^2 (300 \times 10^{-4})^2 + \left(\frac{10^{-1}}{-10^{-8}}\right)^2 (0.3 \times 10^{-4})^2 \tag{B5a}$$

$$Var[PR] = 90{,}000 + 90{,}000 = 180{,}000 \tag{B5b}$$

$$\sigma_{PR} = \sqrt{Var[PR]} = \sqrt{180{,}000} \approx 424 \tag{B5c}$$

The coefficient of variation of the permeability ratio is then:

$$V_{PR} = \left(\frac{\sigma_{PR}}{E[PR]}\right) = \frac{424}{1000} = 42.4\% \tag{B6}$$

Taylor's Series with Numerically Approximated Derivatives

Where derivatives are difficult to precisely calculate, a finite difference approximation can be used, approximating the derivatives using two points, one standard deviation above and below the expected value of each random variable.

The expected value of the function, retaining only first-order terms, is the function of the expected values:

$$E[PR] = PR(E[k_f],E[k_b]) = \frac{1000 \times 10^{-4}}{1 \times 10^{-4}} = 1000 \tag{B7}$$

The variance term

$$Var[PR] = \left(\frac{\partial PR}{\partial k_f}\right)^2 \sigma_{kf}^2 + \left(\frac{\partial PR}{\partial k_b}\right)^2 \sigma_{kb}^2 \tag{B8}$$

can be expressed using finite difference approximations of the derivatives as:

$$Var[PR] = \left(\frac{PR(k_{f+}) - PR(k_{f-})}{2\sigma_{kf}}\right)^2 \sigma_{kf}^2 + \left(\frac{PR(k_{b+}) - PR(k_{b-})}{2\sigma_{kb}}\right)^2 \sigma_{kb}^2 \tag{B9}$$

where $PR(k_{f+})$ refers to the permeability ratio evaluated with k_f taken one standard deviation above the expected value, i.e., $k_{f+} = E[k_f] + \sigma_{kf}$, and the expected value of the other random variables are used. The other terms are developed similarly. Substituting, one obtains

$$Var[PR] = \left(\frac{PR(k_{f+}) - PR(k_{f-})}{2\sigma_{kf}}\right)^2 \sigma_{kf}^2 + \left(\frac{PR(k_{b+}) - PR(k_{b-})}{2\sigma_{kb}}\right)^2 \sigma_{kb}^2$$

$$Var[PR] = \left(\frac{\dfrac{1300 \times 10^{-4}}{1 \times 10^{-4}} - \dfrac{700 \times 10^{-4}}{1 \times 10^{-4}}}{600 \times 10^{-4}}\right)^2 (300 \times 10^{-4})^2$$

$$+ \left(\frac{\dfrac{1000 \times 10^{-4}}{1.3 \times 10^{-4}} - \dfrac{1000 \times 10^{-4}}{0.7 \times 10^{-4}}}{0.60 \times 10^{-4}}\right)^2 (0.30 \times 10^{-4})^2 \tag{B10}$$

$$Var[PR] = 90{,}000 + 108{,}684$$

$$= 198{,}684$$

$$\sigma_{PR} = 445.7$$

The coefficient of variation is then:

$$V_{PR} = \frac{\sigma_{PR}}{E[PR]} = \frac{445.7}{1000} = 44.6\% \tag{B11}$$

Point Estimate Method

Using the point estimate method, the permeability of the foundation is represented by two point estimates and two probability concentrations:

$$
\begin{aligned}
k_{f+} &= E[k_f] + \sigma_{kf} = 1300 \times 10^{-4} cm/\mathrm{sec} \\
k_{f-} &= E[k_f] + \sigma_{kf} = 700 \times 10^{-4} cm/\mathrm{sec} \\
P_{kf+} &= 0.50 \\
P_{kf-} &= 0.50
\end{aligned}
\qquad \text{(B12)}
$$

Likewise, the top blanket permeability is modeled by

$$
\begin{aligned}
k_{b+} &= E[k_b] + \sigma_{kb} = 1.30 \times 10^{-4} cm/\mathrm{sec} \\
k_{b-} &= E[k_b] + \sigma_{kb} = 0.70 \times 10^{-4} cm/\mathrm{sec} \\
P_{kb+} &= 0.50 \\
P_{kb-} &= 0.50
\end{aligned}
\qquad \text{(B13)}
$$

The expected value of the permeability ratio is then

$$
\begin{aligned}
E[PR] &= \sum_{all\ \ combinations} P_{kf\pm} P_{kb\pm} PR_{\pm\pm} \\
E[PR] &= 0.25(PR_{++}) + 0.25(PR_{\pm}) + 0.25(PR_{-+}) + 0.25(PR_{--}) \\
E[PR] &= \frac{1}{4}\left(\frac{1300}{1.3} + \frac{1300}{0.7} + \frac{700}{1.3} + \frac{700}{0.7} \right) \\
&= \frac{1}{4}(1000 + 1857.1 + 701.3 + 1000) \\
&= 1139
\end{aligned}
\qquad \text{(B14)}
$$

Note that the expected value is higher than that found using the Taylor's series method as it picks up some of the nonlinearity of the function which was neglected when the terms above the first order were neglected.

To find the variance, first $E[PR^2]$ is calculated:

$$
\begin{aligned}
E[PR^2] &= 0.25(PR^2_{++}) + 0.25(PR^2_{+-}) + 0.25(PR^2_{-+}) + 0.25(PR^2_{--}) \\
E[PR^2] &= \frac{1}{4}(1000^2 + 1857.1^2 + 701.3^2 + 1000^2) \\
&= 1{,}485{,}200
\end{aligned}
\qquad \text{(B15)}
$$

The variance is then calculated by the identity:

$$\begin{aligned} Var[PR] &= E[PR^2] - (E[PR])^2 \\ &= 1{,}485{,}200 - 1139^2 \\ &= 187{,}879 \end{aligned} \qquad \text{(B16)}$$

and the standard deviation and coefficient of variation are:

$$\begin{aligned} \sigma_{PR} &= \sqrt{187{,}879} = 433 \\ V_{PR} &= \frac{\sigma_{PR}}{E[PR]} = \frac{433}{1139} = 38\% \end{aligned} \qquad \text{(B17)}$$

Note that the estimate of the standard deviation is similar to that for the two Taylor's series methods, but the coefficient of variation drops because the expected value increased.